Iraqi Journal of Statistical Science (20) 2011
The Fourth Scientific Conference of the College of Computer Science & Mathematics
pp [127-145]

Exact Solutions for Some Partial Differential Equations by Using First Integral Method

Faazah L. Hassan Al-Qattrani[**]

ABSTRACT

In this paper, some exact solutions for the convection–diffusion–reaction equation in two dimensions and a nonlinear system of partial differential equations are formally derived by using the first integral method, which are based on the theory of commutative algebra.

Keywords: First integral method; two-dimensional convection–diffusion–reaction equation, nonlinear system of partial differential equation.

الحلول الدقيقة لبَعْض المعادلات التفاضليةِ الجزئية باستعمال

طريقة التكامل ألأول

فائزة لفته حسن القطراني

الخلاصة

في هذا البحث اشتقت صيغة الحلول الدقيقة لمعادلة التفاعل–الانتشار–الحمل ثنائية البعد ونظام غير خطي لمعادلات تفاضلية جزئية باستعمال طريقة التكامل الأول، التي تستند على نظرية الجبر التبادلي.

1. Introduction

In the recent years, the problem of obtaining exact solutions of Nonlinear Partial differential equations (NLPDEs) has attracted attention of many experts, due to the appearance of these equations in many fields such as complex physics phenomena, mechanics, chemistry, engineering and biology etc., and also as a result of the development

[**] Assist. Lecturer\ College of Education\ University of Basrah

Received:1/10 /2011 ____________________Accepted: 21 /12 / 2011

in the field of computer software like Maple or Mathematica, which enables us to perform the complicated and tedious algebraic calculations easily and high efficiency, moreover, the exact solutions of nonlinear equations facilitates the verification of the numerical solutions and aids in the stability analysis of solutions. So, a variety of powerful methods for finding the exact solutions of nonlinear evolution equations had proposed such as, tanh-sech method [9,12,18], extended tanh method [6], sine-cosine method [17], F-expansion method [14], the extended mapping method [19], etc.

Feng [7], proposed a new powerful method, the first integral method for solving Burgers-KdV equation. This method depends on the concept of the theory of commutative algebra[5] and it has many advantages, which is mainly embodied that it could avoid a great deal of complicated and tedious calculations and provide more exact and explicit traveling solitary solutions. So it is considered easier and quicker method than other traditional techniques. Recently this useful method was used widely by many researchers [3,11,13, 15,16].

In the present work, we would like to extend the application of the first integral method to solve the convection–diffusion–reaction equation in two dimensions and the coupled dispersionless system of partial differential equations. The remainder Structure of this paper can be organized as follows: Section 2, is a brief introduction to the first integral method. In section 3, implementing the first integral method, and some new exact solutions for (CDR) and nonlinear system of Partial Differential Equation (PDE) are reported. Finally, in section 4, conclusion for this research is summarized.

2. Basic idea of the first integral method

Consider a general nonlinear PDE in the form

$$P\left(u, u_t, u_x, u_y, u_{tt}, u_{xx}\, u_{yy}\;\; u_{xt}, u_{yt}, u_{xy}, u_{xxx}, \ldots\right) = 0, \qquad (1)$$

where $u(x, y, t)$ is the solution of the equation (1). We use the transformations

$$u(x, y, t) = f(\xi), \;\; \xi = x + \beta y + \alpha t \qquad (2)$$

where β and α are constant. This enables us to use the following changes:

$$\frac{\partial}{\partial t}(\cdot) = \alpha\frac{d}{d\xi}(\cdot);\;\; \frac{\partial}{\partial x}(\cdot) = \frac{d}{d\xi}(\cdot);\;\; \frac{\partial}{\partial y}(\cdot) = \beta\frac{d}{d\xi}(\cdot);\;\; \frac{\partial^2}{\partial x^2}(\cdot) = \frac{d^2}{d\xi^2}(\cdot), \ldots \qquad (3)$$

we use (3) to change the partial differential equation(PDE) (1) to Ordinary Differential Equation(ODE):

$$G(f, f', f'', f''', \ldots) = 0. \qquad (4)$$

Now, we introduce new independent variables $X(\xi) = f(\xi),\ Y(\xi) = f_\xi(\xi)$ which change to a system of ODEs

$$\begin{cases} X' = Y\,, \\ Y' = F\left(X(\xi)\,, Y(\xi)\right). \end{cases} \tag{5}$$

According to the qualitative theory of differential equations [5], if one can find two first integrals to system (5) under the same conditions, then the analytic solutions to (5) can be solved directly. Since, it is difficult to realize this even for a single first integral, because for a given plane autonomous system, there is no general theory telling us how to find its first integrals in a systematic way. A key idea of our approach here to find first integral is to utilize the division theorem. For convenience, first let us recall the division theorem for two variables in the complex domain $\mathbb{C}$.

2.1 Division Theorem [5] Suppose that $P(X,Y)$ and $Q(X,Y)$ are polynomials of two variables X and Y in $\mathbb{C}[X,Y]$ and $P(X,Y)$ is irreducible in $\mathbb{C}[X,Y]$. If $Q(X,Y)$ vanishes at all zero points of $P(X,Y)$, then there exists a polynomial $G(X,Y)$ in $\mathbb{C}[X,Y]$ such that $Q(X,Y) = P(X,Y)\,G(X,Y)$.

3. Application

Example 1

The CDR is practically important because the working equations of many cases fall into this category. Typical examples are the Helmholtz equation for modeling exterior acoustics [2], constitutive equations for modeling the turbulent quantities *k* and *ε* [8].

$$\phi_t + u\phi_x + v\phi_y - k\left(\phi_{xx} + \phi_{yy}\right) + c\phi = 0\,, \tag{6}$$

where *u* and *v* represent the velocity components along the *x* and *y* directions, respectively. Other coefficients involve *k* and *c*, which denote the diffusion coefficient and the reaction coefficient, respectively. For illustrative purposes, all these values are assumed to be constant throughout. For finding exact solutions using (2) and (3), Eq.(6) becomes

$$\alpha f' + uf' + v\beta f' - k(f'' + \beta^2 f'') + cf = 0\ , \tag{7}$$

then

$$f'' = \frac{\alpha + u + v\beta}{k(1+\beta^2)} f' + \frac{c}{k(1+\beta^2)} f\,. \tag{8}$$

Let $X(\xi) = f(\xi),\ Y(\xi) = f_\xi(\xi)$, then equation (8) is equivalent to

$$\begin{cases} X'(\xi) = Y(\xi) \ , \\ Y'(\xi) = \dfrac{\alpha + u + v\beta}{k(1+\beta^2)} Y + \dfrac{c}{k(1+\beta^2)} X. \end{cases} \tag{9}$$

Now, according to the first integral method, we suppose the $X = X(\xi)$ and $Y = Y(\xi)$ are

the nontrivial solutions to (9), and $q(X,Y) = \sum_{i=0}^{m} a_i(X)\, Y^i$ is an irreducible polynomial in $\mathbb{C}[X,Y]$, such that

$$q\big(X(\xi), Y(\xi)\big) = \sum_{i=0}^{m} a_i(X)\, Y^i = 0\,, \tag{10}$$

where $a_i(X) (i = 0,1,2,\dots,m)$ are polynomial of X and $a_m(X) \neq 0$. Eq. (10) is called first integral to equation (9). Due to the Division Theorem, there exists a polynomial $H(X,Y) = g(X) + h(X)Y$ in $\mathbb{C}[X,Y]$, such that

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial X}\frac{dX}{d\xi} + \frac{\partial q}{\partial Y}\frac{dY}{d\xi} = (g(X) + h(X)Y)\left(\sum_{i=0}^{m} a_i(X)\, Y^i\right) \tag{11}$$

in our study will take three cases are clarified as follows:

Case 1

By assuming m =1 in (10). Note that $\frac{dq}{d\xi}$ is a polynomial in X and Y, and $q(X,Y) = 0$ Implies that $\frac{dq}{d\xi} = 0$,

$$\sum_{i=0}^{1} a_i'(X) Y^{i+1} + \sum_{i=0}^{1} i\, a_i(X)\, Y^{i-1}\left(\frac{\alpha + u + v\beta}{k(1+\beta^2)} Y + \frac{c}{k(1+\beta^2)} X\right)$$

$$= (g(X) + h(X)Y)\left(\sum_{i=0}^{1} a_i(X)\, Y^i\right), \tag{12}$$

where prime denotes differentiating with respect to the variable X. On equating the coefficients of $Y^i (i = 2,1,0)$ on both sides of (12), we have

$$a_1'(X) = a_1(X) h(X)\ . \tag{13a}$$

$$a_0'(X) + a_1(X)\left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right) = a_1(X)g(X) + a_0(X)h(X)\,, \tag{13b}$$

$$a_1(X)\left(\frac{c}{k(1+\beta^2)}X\right) = a_0(X)g(X)\,, \tag{13c}$$

since $a_1(X)$ is a polynomial of X, from (13a), we deduce that $a_1(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_1(X) = 1$, and balancing the degrees of $g(X)$, $a_1(X)$ and $a_0(X)$, we conclude that deg $g(X) = 1$ only, suppose that $g(X) = AX + B$, then we find $a_0(X)$ from (13b)

$$a_0(X) = \frac{A}{2}X^2 + \left(B - \frac{\alpha + u + v\beta}{k(1+\beta^2)}\right)X + D, \tag{14}$$

where D is an arbitrary integration constant. Substituting $a_1(X)$, $a_0(X)$ and $g(X)$ in (13c) and setting all the coefficients of powers X to be zero. Then we obtain a system of nonlinear algebraic equations

$$BD = 0, \qquad B^2 + AD - \frac{Bu}{k(1+\beta^2)} - \frac{B\alpha}{k(1+\beta^2)} - \frac{Bv\beta}{k(1+\beta^2)} - \frac{c}{k(1+\beta^2)} = 0\,,$$
$$\frac{3}{2}AB - \frac{Au}{k(1+\beta^2)} - \frac{A\alpha}{k(1+\beta^2)} - \frac{Av\beta}{k(1+\beta^2)} = 0\,, \qquad \frac{A^2}{2} = 0, \tag{15}$$

solving equation (15), we can obtain that

$$A = 0, D = 0, B = \frac{u + \alpha + v\beta - \sqrt{(-u - \alpha - v\beta)^2 + 4c(k + k\beta^2)}}{2(k + k\beta^2)}\,, \tag{16a}$$

$$A = 0, D = 0, B = \frac{u + \alpha + v\beta + \sqrt{(-u - \alpha - v\beta)^2 + 4c(k + k\beta^2)}}{2(k + k\beta^2)}\,, \tag{16b}$$

using (16a) and (16b) in (10), we obtain

$$Y = -\left(\frac{u + \alpha + v\beta + \sqrt{(u + \alpha + v\beta)^2 + 4ck(1+\beta^2)}}{2k(1+\beta^2)}\right)X, \tag{17a}$$

and

$$Y = -\left(-\frac{u + \alpha + v\beta - \sqrt{(u + \alpha + v\beta)^2 + 4ck(1+\beta^2)}}{2k(1+\beta^2)}\right)X, \tag{17b}$$

respectively. Combining equation (17)with (9), we obtain the exact solutions of Eq. (8) as follows:

$$f(\xi)=e^{\frac{\left(u+\alpha+v\beta+\sqrt{(u+\alpha+v\beta)^2+4ck(1+\beta^2)}\right)\xi}{2k(1+\beta^2)}}C_1, \quad (18a)$$

$$f(\xi)=e^{\frac{\left(u+\alpha+v\beta-\sqrt{(u+\alpha+v\beta)^2+4ck(1+\beta^2)}\right)\xi}{2k(1+\beta^2)}}C_1, \quad (18b)$$

where C_1 is an arbitrary integration constant. Then the exact solution to (6) can be written as

$$\phi(x,y,t)=e^{\frac{\left(u+\alpha+v\beta+\sqrt{(u+\alpha+v\beta)^2+4ck(1+\beta^2)}\right)(x+\beta y+\alpha t)}{2k(1+\beta^2)}}C_1, \quad (19a)$$

$$\phi(x,y,t)=e^{\frac{\left(u+\alpha+v\beta-\sqrt{(u+\alpha+v\beta)^2+4ck(1+\beta^2)}\right)(x+\beta y+\alpha t)}{2k(1+\beta^2)}}C_1. \quad (19b)$$

Case 2

We assume that $m=2$ in (10), and $q(X,Y)=0$ this implies that $\frac{dq}{d\xi}=0$,

$$\sum_{i=0}^{2} a_i'(X)Y^{i+1}+\sum_{i=0}^{2} i\,a_i(X)\,Y^{i-1}\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}Y+\frac{c}{k(1+\beta^2)}X\right)$$
$$=\left(g(X)+h(X)Y\right)\left(\sum_{i=0}^{2} a_i(X)\,Y^i\right), \quad (20)$$

on equating the coefficients of $Y^i\ (i=3,2,1,0)$ on both sides of (20), we have

$$a_2'(X)=a_2(X)h(X)\ , \quad (21a)$$

$$a_1'(X)+2a_2(X)\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)=a_2(X)g(X)+a_1(X)h(X)\,, \quad (21b)$$

$$a_0'(X)+2a_2(X)\left(\frac{c}{k(1+\beta^2)}X\right)+a_1(X)\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)$$
$$=a_1(X)g(X)+a_0(X)h(X), \quad (21c)$$

$$a_1(X)\left(\frac{c}{k(1+\beta^2)}X\right)=a_0(X)g(X)\,, \quad (21d)$$

since $a_2(X)$ is a polynomial of X, from (21a), we deduce that $a_2(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_2(X) = 1$, and balancing the degrees of $g(X)$, $a_1(X)$ and $a_0(X)$, we conclude that deg $g(X) = 1$ only, suppose that $g(X) = Ax+B$, then we find $a_1(X)$, and $a_0(X)$ From (25b,c)

$$a_1(X) = D + \left(B - 2\left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right)\right)X + \frac{1}{2}AX^2 , \qquad (22)$$

where D is an arbitrary integration constant. Substituting $a_2(X)$,$a_1(X)$and $g(X)$ in (21c)

$$a_0(X) = E + \left(Bd - D\left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right)\right)X + \frac{1}{2}\left(B^2 + AD - 2\left(\frac{c}{k(1+\beta^2)}\right) - 3B\left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right) + 2\left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right)^2\right)X^2 + \frac{1}{3}\left(\frac{3AB}{2} - \frac{5A\left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right)}{2}\right)X^3 + \frac{1}{8}A^2X^4, \qquad (23)$$

where E is an arbitrary integration constant. Substituting $a_0(X)$, $a_1(X)$and $g(X)$ in (21d) and setting all the coefficients of powers X to be zero. Then we obtain a system of nonlinear algebraic equations and by solving it, we get

$$E = 0, A = 0, D = 0, \quad B = \left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right), \qquad (24a)$$

$$E = 0, A = 0, D = 0, B = \left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right) - \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right) + \left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right)^2}, \qquad (24b)$$

$$E = 0, A = 0, D = 0, B = \left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right) + \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right) + \left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right)^2}, \qquad (24c)$$

using (24a) in (10), we obtain

$$Y = \frac{1}{2}\left(\frac{\alpha + u + v\beta}{k(1+\beta^2)} - \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right) + \left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right)^2}\right)X, \qquad (25a)$$

$$Y = \frac{1}{2}\left(\frac{\alpha + u + v\beta}{k(1+\beta^2)} + \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right) + \left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right)^2}\right)X, \qquad (25b)$$

respectively. Combining equations (25) with (9), we obtain the exact solutions of Eq.(8) as follows:

$$f(\xi) = e^{\frac{1}{2}\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)} - \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right)+\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2}\right)\xi} C_1\,, \tag{26a}$$

$$f(\xi)= e^{\frac{1}{2}\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)} + \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right)+\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2}\right)\xi} C_1. \tag{26b}$$

where C_1 is an arbitrary integration constant. Then the exact solution to (6) can be written as

$$\phi(x,y,t) = e^{\frac{1}{2}\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)} - \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right)+\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2}\right)(x+\beta y+\alpha t)} C_1, \tag{27a}$$

$$\phi(x,y,t) = e^{\frac{1}{2}\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)} + \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right)+\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2}\right)(x+\beta y+\alpha t)} C_1. \tag{27b}$$

Similarly, as for the cases of (24b) and (24c) the exact solutions are respectively

$$\phi(x,y,t) = e^{\frac{1}{2}\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)} + \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right)+\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2}\right)(x+\beta y+\alpha t)} C_1, \tag{28}$$

$$\phi(x,y,t) = e^{\frac{1}{2}\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)} - \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right)+\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2}\right)(x+\beta y+\alpha t)} C_1. \tag{29}$$

Case 3

Finally, we assume $m = 3$ in (10). By the same procedure in the last cases

$$\sum_{i=0}^{3} a_i'(X)Y^{i+1} + \sum_{i=0}^{3} i\, a_i(X)\, Y^{i-1}\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}Y + \frac{c}{k(1+\beta^2)}X\right) = (g(X)+h(X)Y)\left(\sum_{i=0}^{3} a_i(X)\,Y^i\right). \tag{30}$$

On equating the coefficients of Y^i $(i = 4,3,2,1,0)$ on both sides of (30), we have

$$a_3'(X) = a_3(X)h(X)\ , \tag{31a}$$

$$a_2'(X) + 3a_3(X)\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right) = a_3(X)g(X) + a_2(X)h(X)\,, \tag{31b}$$

$$a_1'(X) + 3a_3(X)\left(\frac{c}{k(1+\beta^2)}X\right) + 2a_2(X)\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)$$
$$= a_2(X)g(X) + a_1(X)h(X), \qquad (31c)$$

$$a_0'(X) + 2a_2(X)\left(\frac{c}{k(1+\beta^2)}X\right) + a_1(X)\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)$$
$$= a_1(X)g(X) + a_0(X)h(X), \tag{31d}$$

$$a_1(X)\left(\frac{c}{k(1+\beta^2)}X\right) = a_0(X)g(X)\,, \tag{31e}$$

since $a_3(X)$ is a polynomial of X, from (31a), we deduce that $a_3(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_3(X) = 1$, and balancing the degrees of $g(X)$, $a_2(X)$, $a_1(X)$ and $a_0(X)$, we conclude that deg $g(X) = 1$ only, suppose that $g(X) = AX+B$, then we find $a_2(X)$, $a_1(X)$ and $a_0(X)$ from (30 b, c, d)

$$a_2(X) = D + \left(B - 3\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)\right)X + \frac{1}{2}AX^2\,, \tag{32}$$

$$a_1(X) = E + \left(B\,D - 2D\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)\right)X$$
$$+ \frac{1}{2}\left(B^2 + A\,D - 3\left(\frac{c}{k(1+\beta^2)}\right) - 5B\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)\right.$$
$$\left. + \;6\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2\right)X^2 + \frac{1}{3}\left(\frac{3AB}{2} - 4A\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)\right)X^3 + \frac{1}{8}A^2X^4,$$

(33)

$$
\begin{aligned}
a_0(X) = F &+ \left(BE - E\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)\right)X \\
&+ \frac{1}{2}\left(B^2 D - 2D\left(\frac{c}{k(1+\beta^2)}\right) + AE - 3BD\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)\right. \\
&\left. + 2D\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2\right)X^2 \\
&+ \frac{1}{3}\left(\frac{B^3}{2} + \frac{3ABD}{2} - \frac{7B\left(\frac{c}{k(1+\beta^2)}\right)}{2} - 3B^2\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)\right. \\
&- \frac{5AD\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)}{2} + \frac{15\left(\frac{c}{k(1+\beta^2)}\right)\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)}{2} \\
&\left. + \frac{11B\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2}{2} - 3\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^3\right)X^3 \\
&+ \frac{1}{4}\left(AB^2 + \frac{A^2 D}{2} - \frac{5A\left(\frac{c}{k(1+\beta^2)}\right)}{2} - \frac{13AB\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)}{3}\right. \\
&\left. + \frac{13A\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2}{3}\right)X^4 + \frac{1}{5}\left(\frac{5A^2 B}{8} - \frac{35A^2\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)}{24}\right)X^5 \\
&+ \frac{1}{48}A^3 X^6\ , \qquad (34)
\end{aligned}
$$

where D, E and F are an arbitrary integration constants. substituting $a_0(X)$, $a_1(X)$and $g(X)$ in (31e) and setting all the coefficients of powers X to be zero. Then we obtain a system of nonlinear algebraic equations

$$F = 0, E = 0, A = 0, D = 0,$$

$$B = \frac{3}{2}\left(\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right) - \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right) + \left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2}\right), \qquad (35a)$$

$F = 0, E = 0, A = 0, D = 0,$

$$B = \frac{1}{2}\left(3\left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right) - \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right) + \left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right)^2} \right), \tag{35b}$$

$F = 0, E = 0, A = 0, D = 0,$

$$B = \frac{3}{2}\left(\left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right) + \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right) + \left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right)^2} \right), \tag{35c}$$

$F = 0, E = 0, A = 0, D = 0,$

$$B = \frac{1}{2}\left(3\left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right) + \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right) + \left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right)^2} \right), \tag{35d}$$

using (35a) in (10), we obtain

$$Y = \frac{1}{2}\left(\left(\left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right) + \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right) + \left(\frac{\alpha + u + v\beta}{k(1+\beta^2)}\right)^2} \right)\right), \tag{36}$$

respectively. Combining equations (36) with (9), we obtain the exact solutions of Eq.(8) as follows:

$$f(\xi) = e^{\frac{1}{2}\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)} + \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right)+\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2}\right)\xi} C_1, \tag{37}$$

where C_1 is an arbitrary integration constant and Then the exact solution to (6) can be written as

$$\phi(x,y,t) = e^{\frac{1}{2}\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)} + \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right)+\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2}\right)(x+\beta y+\alpha t)} C_1 . \tag{38}$$

Similarly, as for the cases of (35b), (35c) and (35d) the exact solutions are respectively

$$\phi(x,y,t) = e^{\frac{\alpha+u+v\beta}{2k(1+\beta^2)} - \frac{2c}{k(1+\beta^2)\sqrt{4\left(\frac{c}{k(1+\beta^2)}\right)+\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2}} - \frac{(\alpha+u+v\beta)^2}{2(k(1+\beta^2))^2\sqrt{4(\frac{c}{k(1+\beta^2)})+(\frac{\alpha+u+v\beta}{k(1+\beta^2)})^2}} + C_1} \tag{39}$$

$$\phi(x,y,t) = e^{\frac{\alpha+u+v\beta}{2k(1+\beta^2)} + \frac{2c}{k(1+\beta^2)\sqrt{4\left(\frac{c}{k(1+\beta^2)}\right)+\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2}^{2}} + \frac{(\alpha+u+v\beta)^2}{2(k(1+\beta^2))^2\sqrt{4\left(\frac{c}{k(1+\beta^2)}\right)+\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)}^{2}} + C_1} \tag{40}$$

$$\phi(x,y,t) = e^{\frac{\alpha+u+v\beta}{2k(1+\beta^2)} + \frac{2c}{k(1+\beta^2)\sqrt{4\left(\frac{c}{k(1+\beta^2)}\right)+\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2}^{2}} + \frac{(\alpha+u+v\beta)^2}{2(k(1+\beta^2))^2\sqrt{4\left(\frac{c}{k(1+\beta^2)}\right)+\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)}^{2}} + C_1} \tag{41}$$

$$\phi(x,y,t) = e^{\frac{1}{2}\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)} - \sqrt{4\left(\frac{c}{k(1+\beta^2)}\right)+\left(\frac{\alpha+u+v\beta}{k(1+\beta^2)}\right)^2}\right)(x+\beta y+\alpha t)} \quad C_1 . \tag{42}$$

where C_1 is an arbitrary integration constant. These solutions are all new exact solutions

Example 2

The new coupled equation has been first presented by Konno and Oono where solved by using the Inverse Scattering Transform (IST) method [10] and also extended mapping method [19]. The integrability properties of the coupled dispersionless system was employed by the Painlevé test [1].

$$u_{xt} - 2uv = 0 , \tag{43}$$

$$v_t + 2u\,u_x = 0 ,$$

introducing the following transformations

$$u(x,t) = f(\xi),$$

$$u(x,t) = f(\xi), \tag{44}$$

where, $\xi = x + \alpha t,$ the system (43) reduces to

$$\alpha f'' - 2fg = 0 ,$$

$$\alpha g' + (f^2)'' = 0, \tag{45}$$

integrating the second segment of equation (45) with respect to ξ yields

$$g = \frac{c - f^2}{\alpha}, \tag{46}$$

substituting equation (46) into the first segment of equation (45) yields

$$f'' = \frac{2c}{\alpha^2} f - \frac{2}{\alpha^2} f^3, \tag{47}$$

Let $X(\xi) = f(\xi),\ Y(\xi) = f_\xi(\xi)$, then equation (47) is equivalent to

$$\begin{cases} X'(\xi) = Y(\xi)\ , \\ Y'(\xi) = \dfrac{2c}{\alpha^2} X(\xi) - \dfrac{2}{\alpha^2} X(\xi)^3 , \end{cases} \tag{48}$$

according to the first integral suppose that *X* = *X*(*ξ*) and *Y* = *Y*(*ξ*) are the nontrivial

solutions to (48), and $q(X, Y) = \sum_{i=0}^{m} a_i(X)\, Y^i$ is an irreducible polynomial in the complex domain *in* $\mathbb{C}\,[X, Y]$ such that

$$q\big(X(\xi), Y(\xi)\big) = \sum_{i=0}^{m} a_i(X(\xi))\, Y(\xi)^i = 0\,, \tag{49}$$

where $a_i(X)(i = 0,1,2, \dots, m)$ are polynomial of X and $a_m(X) \neq 0$. Eq.(48) is called first integral to equation (48). By the Division Theorem, there exists a polynomial $H(X, Y) = g(X) + h(X)Y$ in $\mathbb{C}\,[X, Y]$ such that

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial X}\frac{dX}{d\xi} + \frac{\partial q}{\partial Y}\frac{dY}{d\xi} = \big(g(X) + h(X)Y\big)\left(\sum_{i=0}^{m} a_i(X)\, Y^i\right). \tag{50}$$

<u>Case 1</u>

Assuming *that m* =1 in (49). Note that $\frac{dq}{d\xi}$ is a polynomial in *X* and *Y*, and $q(X, Y) = 0$ implies that $\frac{dq}{d\xi} = 0$,

$$\sum_{i=0}^{1} a_i'(X) Y^{i+1} + \sum_{i=0}^{1} i\, a_i(X)\, Y^{i-1}\left(\frac{2c}{\alpha^2} X - \frac{2}{\alpha^2} X^3\right) = \big(g(X) + h(X)Y\big)\left(\sum_{i=0}^{1} a_i(X)\, Y^i\right), \tag{51}$$

by comparison with the coefficients of Y^i $(i = 2,1,0)$ from both sides of (51), we have

$$a_1'(X) = a_1(X)h(X) \ , \tag{52a}$$

$$a_0'(X) = a_1(X)g(X) + a_0(X)h(X) \, , \tag{52b}$$

$$a_1(X)\left(\frac{2c}{\alpha^2}X - \frac{2}{\alpha^2}X^3\right) = a_0(X)g(X) \, , \tag{52c}$$

since $a_1(X)$ is a polynomial of X, from (51a), we deduce that $a_1(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_1(X) = 1$, and balancing the degrees of $g(X)$, $a_1(X)$ and $a_0(X)$, we conclude that deg $g(X) = 1$ only. Suppose that $g(X) = AX + B$, then we find $a_0(X)$ from (52b)

.

$$a_0(X) = \frac{A}{2}X^2 + BX + D, \tag{53}$$

where D is an arbitrary integration constant. Substituting $a_0(X)$and $g(X)$ in (52c) and setting all the coefficients of powers X to be zero. Then we obtain a system of nonlinear algebraic equations

$$BD = 0, \qquad B^2 + AD = \frac{2c}{\alpha^2}, \qquad \frac{3}{2}AB = 0 \, , \qquad \frac{A^2}{2} = -\frac{2}{\alpha^2}, \tag{54}$$

solving the last algebraic equations, we obtain

$$D = -\frac{i\,c}{\alpha}, \quad A = \frac{2\,i}{\alpha}, \qquad B = 0\, , \tag{55a}$$

$$D = \frac{i\,c}{\alpha}, \quad A = -\frac{2\,i}{\alpha}, B = 0\, , \tag{55b}$$

using (55a) and (55b) in (49), we obtain

$$Y = \frac{i\,(c\ - X^2)}{\alpha}, \tag{56a}$$

and

$$Y = \frac{i\,(-c\ + X^2)}{\alpha}, \tag{56b}$$

respectively. Combining equation (56)with (48), we obtain the exact solutions of Eq. (47) as follows:

$$f(\xi) = \sqrt{c}\ \ \tanh\left(\frac{i\sqrt{c}\ \ \xi + \sqrt{c}\ \ \alpha\, C_1}{\alpha}\right), \tag{57a}$$

$$f(\xi) = \sqrt{c}\ \ \tanh\left(\frac{-\,i\sqrt{c}\ \ \xi + \sqrt{c}\ \ \alpha\, C_1}{\alpha}\right), \tag{57b}$$

where C_1 is an arbitrary integration constant. Then the exact solution to system (43) can be written as

$$u(x,t) = \sqrt{c}\ \tanh\left(\frac{i\sqrt{c}\ (x+\alpha t)+\sqrt{c}\ \alpha C_1}{\alpha}\right),$$

$$v(x,t) = \frac{c\ \mathrm{Sec}\left(\frac{\sqrt{c}\ ((x+\alpha t)-i\,\alpha\, C_1)}{\alpha}\right)^2}{\alpha}, \tag{58a}$$

and

$$u(x,t) = \sqrt{c}\ \tanh\left(\frac{-i\sqrt{c}\ (x+\alpha t)+\sqrt{c}\ \alpha C_1}{\alpha}\right),$$

$$v(x,t) = \frac{c\ \mathrm{Sec}\left(\frac{\sqrt{c}\ ((x+\alpha t)+i\,\alpha\, C_1)}{\alpha}\right)^2}{\alpha}. \tag{58b}$$

Case 2

Assuming that m =2, by comparison with the coefficients of $Y^i\ (i=3,2,1,0)$ from both sides of (51), we have

$$a_2'(X) = a_2(X)h(X)\ , \tag{59a}$$

$$a_1'(X) = a_2(X)g(X) + a_1(X)h(X)\ , \tag{59b}$$

$$a_0'(X) + 2a_2(X)\left(\frac{2c}{\alpha^2}X - \frac{2}{\alpha^2}X^3\right) = a_1(X)g(X) + a_0(X)h(X)\ , \tag{59c}$$

$$a_1(X)\left(\frac{2c}{\alpha^2}X - \frac{2}{\alpha^2}X^3\right) = a_0(X)g(X)\ , \tag{59d}$$

since $a_2(X)$ is a polynomial of X, from (59a), we deduce that $a_2(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_2(X) = 1$, and balancing the degrees of $g(X)$, $a_1(X)$, $a_2(X)$ and $a_0(X)$, we conclude that deg $g(X) = 1$ only. Suppose that $g(X) = AX + B$, then from (59b) and (59c)we find $a_1(X)$ and $a_2(X)$ as follows:

. $$a_2(X) = \frac{A}{2}X^2 + BX + D, \tag{60}$$

$$a_1(X) = E + BDX + \left(\frac{B^2}{2} + \frac{1}{2}AD - \frac{2c}{\alpha^2}\right)X^2 + \frac{1}{2}ABX^3 + \left(\frac{A^2}{8} + \frac{1}{\alpha^2}\right)X^4 \tag{61}$$

where D *and* E are an arbitrary integration constants. Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ in (59d) and setting all the coefficients of powers X to be zero. Then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$E = -\frac{c^2}{\alpha^2}, \qquad D = -\frac{2ic}{\alpha}, \qquad B = 0, \qquad A = \frac{4i}{\alpha}, \tag{62a}$$

$$E = -\frac{c^2}{\alpha^2}, \qquad D = \frac{2ic}{\alpha}, \qquad B = 0, \qquad A = -\frac{4i}{\alpha}, \tag{62b}$$

using (62a) and (62b) in (49), we obtain

$$Y = \frac{i(c - X^2)}{\alpha}, \tag{63a}$$

and

$$Y = -\frac{i(c - X^2)}{\alpha}, \tag{63b}$$

Combining equation (63)with (48), we obtain the exact solutions of Eq. (47) as follows:

$$f(\xi) = \sqrt{c}\ \tanh\left(\frac{i\sqrt{c}\ \xi + \sqrt{c}\ \ \alpha C_1}{\alpha}\right), \tag{64a}$$

$$f(\xi) = \sqrt{c}\ \tanh\left(\frac{-i\sqrt{c}\ \xi + \sqrt{c}\ \ \alpha C_1}{\alpha}\right), \tag{64b}$$

where C_1 is an arbitrary integration constant. Then we get the exact solution of Eq.(43) as

$$u(x,t) = \sqrt{c}\ \tanh\left(\frac{i\sqrt{c}\ (x+\alpha t) + \sqrt{c}\ \ \alpha C_1}{\alpha}\right),$$
$$v(x,t) = \frac{c\ \operatorname{Sec}\left(\frac{\sqrt{c}\ \left((x+\alpha t) - i\,\alpha\, C_1\right)}{\alpha}\right)^2}{\alpha}, \tag{65a}$$

and

$$u(x,t) = \sqrt{c}\ \tanh\left(\frac{-i\sqrt{c}\ (x+\alpha t) + \sqrt{c}\ \ \alpha C_1}{\alpha}\right),$$
$$v(x,t) = \frac{c\ \operatorname{Sec}\left(\frac{\sqrt{c}\ \left((x+\alpha t) + i\,\alpha\, C_1\right)}{\alpha}\right)^2}{\alpha}. \tag{65b}$$

Case 3

Assuming that $m = 3$, by equating the coefficients of $Y^i\ (i = 4,3,2,1,0)$ from both sides of (51), we have

$$a_3'(X) = a_3(X)h(X)\ , \tag{66a}$$

$$a_2'(X) = a_2(X)g(X) + a_1(X)h(X)\ , \tag{66b}$$

$$a_1'(X) + 3a_3(X)\left(\frac{2c}{\alpha^2}X - \frac{2}{\alpha^2}X^3\right) = a_2(X)g(X) + a_1(X)h(X)\ , \tag{66c}$$

$$a_0'(X) + 2a_2(X)\left(\frac{2c}{\alpha^2}X - \frac{2}{\alpha^2}X^3\right) = a_1(X)g(X) + a_0(X)h(X)\ , \tag{66d}$$

$$a_1(X)\left(\frac{2c}{\alpha^2}X - \frac{2}{\alpha^2}X^3\right) = a_0(X)g(X)\ , \tag{66e}$$

by the same method in the last cases we obtain the following solutions

$$u(x,t) = \sqrt{c}\ \tanh\left(\frac{i\sqrt{c}\ (x+\alpha t) + \sqrt{c}\ \ \alpha\, C_1}{\alpha}\right),$$

$$v(x,t) = \frac{c\ \text{Sec}\left(\frac{\sqrt{c}\ \left((x+\alpha t) - i\,\alpha\, C_1\right)}{\alpha}\right)^2}{\alpha}. \tag{67}$$

$$u(x,t) = \sqrt{c}\ \tanh\left(\frac{-\,i\sqrt{c}\ (x+\alpha t) + \sqrt{c}\ \ \alpha\, C_1}{\alpha}\right),$$

$$v(x,t) = \frac{c\ \text{Sec}\left(\frac{\sqrt{c}\ \left((x+\alpha t) + i\,\alpha\, C_1\right)}{\alpha}\right)^2}{\alpha}\ . \tag{68}$$

$$u(x,t) = -\frac{4\,i\sqrt{2}\ \sqrt{c}\ e^{\frac{\sqrt{2}\ \sqrt{c}\ \left((x+\alpha\, t) + i\ \alpha\, C_1\right)}{\alpha}}}{-4 + e^{\frac{2\sqrt{2}\ \sqrt{c}\ \left((x+\alpha\, t) + i\ \alpha\, C_1\right)}{\alpha}}}\ , \tag{69}$$

$$v(x,t)=\frac{c\left(16+24\ e^{\frac{2\sqrt{2}\ \sqrt{c}\ ((x+\alpha t)+i\ \alpha C_1)}{\alpha}}+e^{\frac{4\sqrt{2}\ \sqrt{c}\ ((x+\alpha t)+i\ \alpha C_1)}{\alpha}}\right)}{\left(-4+e^{\frac{2\sqrt{2}\ \sqrt{c}\ ((x+\alpha t)+i\ \alpha C_1)}{\alpha}}\right)^2\alpha}.$$

$$u(x,t)=-\frac{4\,i\sqrt{2}\ \sqrt{c}\ e^{\frac{\sqrt{2}\ \sqrt{c}\ ((x+\alpha t)+i\ \alpha C_1)}{\alpha}}}{-4\ e^{\frac{2\sqrt{2}\ \sqrt{c}\ (x+\alpha t)}{\alpha}}+e^{2\ i\sqrt{2}\ \sqrt{c}\ C_1}},$$

$$v(x,t)=\frac{c+\frac{32\,c\,e^{\frac{2\sqrt{2}\ \sqrt{c}\ ((x+\alpha t)+i\ \alpha C_1)}{\alpha}}}{\left(-4\ e^{\frac{2\sqrt{2}\ \sqrt{c}\ (x+\alpha t)}{\alpha}}+e^{2\,i\sqrt{2}\ \sqrt{c}\ C_1}\right)^2}}{\alpha}. \qquad (70)$$

Where C_1 is an arbitrary integration constant. These solutions are all new exact solutions.

4. Conclusions

We applied the first integral method for finding some new exact solutions for two-dimensional convection–diffusion–reaction equation and nonlinear system. This method was proved the applicability and effectiveness for solving these equations.

5. References

[1] Alagesan, T., Chung, Y. and Wakkeeran, K., Bäcklund Transformation and Soliton Solutions for The Coupled Dispersionless Equations , Chao. Solit. and Frac. 21, (2004), 63–67.

[2] Babus, k., Ihlenburg, F., Paik, E. and Sauter, S., A Generalized Finite Element Method for Solving The Helmholtz Equation in Two Dimensions with Minimal Pollution, Comput. Meth. in App. Mech. and Eng., 128, (1995), 325–359.

[3] Davood, R. et. al., The First Integral Method for Solving Maccari's System, App. Math., 2, (2011), 258-263.

[4] Deng, X. and HAN, L., Exact Peaked Wave Solution of The Osmosis *K*(2, 2) Equation, Turk. J. Phys., No.33, (2009), 179 – 184.

[5] Ding, T. and Li, C., Ordinary Differential Equations, Peking University Press, Peking, (1996).

[6] El-Wakil, S. and Abdou, M., New Exact Travelling Wave Solutions Using Modified Extended Tanh-Function Method, Chao. Solit. Frac., 31, (2007), No. 4, 840-852.

[7] Feng, Z., On Explicit Exact Solutions to The Compound Burgers-KdV Equation, Phys. Lett. A. 308, (2002), 173-178.

[8] Ilinca, F. and Pelletier, D., Positivity Preservation and Adaptive Solution for The k^2 Model of Turbulence, AIAA J. 36(1), 44 (1998).
[9] Khater, A., Malfliet, W., Callebaut, D. and Kamel, E., The Tanh Method, a Simple ransformation and Exact Analytical Solutions for Nonlinear Reaction–Diffusion equations, Chao. Solit. Frac., 14, (2002) No. 3, 513-522.
[10] Kakuhata, H. and Konno, K., Canonical Formulation of A Generalized Coupled Dispersionless System, J. Phys. A: Math. Gen., 30, (1997) No.12
[11] Li, H. and Guo, Y., New Exact Solutions to The Fitzhugh-Nagumo Equation, App. Math. and Compu., 180, (2006), No. 2, 524-528.
[12] Malfliet, W., Solitary Wave Solutions of Nonlinear Wave Equations, Am. J. of Phys., 60, (1992), No. 7, 650-654.
[13] Sharma,P. and Kushel,O., The First Integral Method for Huxley Equation, Int. J. of Nonl. Sci., 10, (2010), No. 1, 46-52.
[14] Sheng, Z., The Periodic Wave Solutions for The (2+1) -Dimensional Konopelchenko Dubrovsky Equations, Chao. Solit. Frac., 30, (2006), 1213-1220.
[15] Soliman, A. and Raslan, K., First Integral Method for The Improved Modified KdV Equation, Int. J. of Nonl. Sci. 8, (2009), No.1, pp.11-18.
[16] Taghizadeh, N., Mirzazadeh, M. and Farahrooz, F., New Exact Solutions of Some onlinear Partial Differential Equations by The First Integral Method, Int. J. of Applic. and Appl. Math., 05, (2010), No. 10, 1543– 1553.
[17] Wazwaz, A., The Sine-Cosine Method for Obtaining Solutions with Compact and Noncompact structures, Appl. Math. Comput, 159, (2004), No. 2, 559-576.
[18] Wazwaz, A., Two Reliable Methods for Solving Variants of The KdV Equation with Compact and Noncompact Structures ", Chao. Solit. Frac., 28, (2006), No.2, 454-462.
[19] Peng, Y. and Krishnan, E., Exact Travelling Wave Solutions for a Class of Nonlinear Partial Differential Equations, Int. J. Pure & Appl. Math. Sci., 3, (2006), No.1, 11-20.