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ON MULTIVALENT HARMONIC MEROMORPHIC FUNCTIONS INVOLVING HYPERGEOMETRIC FUNCTIONS

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Abstract

In this paper we introduce a subclass of multivalent harmonic meromorphic functions defined in the exterior of the unit disk by using generalize hypergeometric functions. We derived sufficient coefficient conditions and shown to be also necessary for this subclass by putting certain restrictions on the coefficients, distortion theorem, extreme points and other interesting results are also investigated.

KeyWords: Multivalent functions, Meromorphic functions, Harmonic functions, Distortion theorem, Starlike functions, Hypergemetric functions.

AMS Subject Classication: Secondary30C45.

1. Introduction

A continuous function $f = u + iv$ is a complex – valued harmonic function in a domain $D \subset C$ if both u and v are real harmonic in D. We can write in any simply connected domain D, $f = h + \overline{g}$, where h and g are analytic in D.

Hengartner and Schober[3],considered harmonic sense preserving univalent mappings defined on $\overline{U} = \{z : |z| > 1\}$ that map ∞ to ∞ and represented by

$f(z) = h(z) + \overline{g(z)} + A\log|z|$ where $h(z) = \alpha z + \sum_{n=0}^{\infty} a_n z^{-n}$, $g(z) = \beta z + \sum_{n=1}^{\infty} b_n z^{-n}$ **are** analytic in $\overline{U}$ and $|\alpha| > |\beta| \geq 0, A \in C$, further ,let us denoted the family $\sum_p(H)$ consisting of all harmonic sense-preserving multivalent meromorphic mapping

$$f(z) = h(z) + \overline{g(z)} \qquad (1)$$

Where

$$h(z) = z^p + \sum_{n=1}^{\infty} a_{n+p-1} z^{-(n+p-1)} \quad , \; g(z) = \sum_{n=1}^{\infty} b_{n+p-1} z^{-(n+p-1)}, |b_p| < 1, |z| > 1 \qquad (2)$$

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For $0 \le \alpha < 2(1-k), 0 \le \lambda \le 1, \frac{1}{2} \le k < 1, 0 \le \theta < 1, 0 \le \xi < 1 and z = re^{i\beta}; 1 < r < \infty; \beta, \xi, \theta$ and α are real .

Furthermore, for real or complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_q$ and $\beta_1, \beta_2, \ldots, \beta_s (\beta_j \ne 0, -1, -2, -3, \ldots; j = 1, 2, .., s)$, we define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by

$$ {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_q)_n}{(\beta_1)_n \ldots (\beta_s)_n} \frac{z^n}{n!} \quad (3) $$

$$ (q \le s+1; q, s \in N_0 = N \cup \{0\}; z \in \mathcal{U}), $$

Where $(x)_k$ is the pochhammer symbol, defined in term of Gamma function Γ ,by

$$ (x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}. $$

Corresponding to a function $H_{p,\mu}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z^p \, {}_qF_s \, (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$, we consider a linear operator $H_{p,\mu}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ defined by the convolution

$$ H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) * H_{p,\mu}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \frac{z^p}{(1-z)^{\mu+p}} . (\mu > -p) $$

Let $H_{p,q,s}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) : \sum_p (H) \to \sum_p (H)$ defined by

$$ H_{p,q,s}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) = H_{p,\mu}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z) \quad (4) $$
$$ (\alpha_i, \beta_j \ne 0, -1, -2, -3, \ldots; i = 1, 2, .., q, j = 1, .., s, \mu > -p; f \in \sum_p (H); z \in \overline{U}) $$

For notational simplicity, we use shorter notation

$$ H^{\mu}_{p,q,s}(\alpha_1) = H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) $$

Thus, from(4) we deduce that after simple calculations, we obtain

$$ z(H^{\mu}_{p,q,s}(\alpha_1) f(z)) = z^p + \sum_{n=1}^{\infty} \frac{(\mu+p)_{n+p} (\beta_1)_{n+p} \ldots (\beta_s)_{n+p}}{(\alpha_1)_{n+p} \ldots (\alpha_q)_{n+p}} a_{n+p-1} z^{-n+p-1}. \quad \textbf{(5)} $$

We note that the linear operator $H^{\mu}_{p,q,s}$ is closely related to the Choi-Saigo-Srivastava Operator [2]. In view of relationship (5)for harmonic function $f = h + \overline{g}$ given by (1), we define the operator

$$ H^{\mu}_{p,q,s} f(z) = H^{\mu}_{p,q,s} h(z) + \overline{H^{\mu}_{p,q,s} g(z)}. $$

We introduce the subclass $GO_H(\beta, \lambda, k, p)$ consisting of all functions f satisfying

$$\operatorname{Re}\left\{(1+e^{i\theta})\frac{z(H^{\mu}_{p,q,s}f(z))'/H^{\mu}_{p,q,s}f(z)}{\lambda z(H^{\mu}_{p,q,s}f(z))'/H^{\mu}_{p,q,s}f(z)+(1-p\lambda)}-pk(1+e^{i\xi})\right\}\geq p\alpha \qquad (6)$$

.The special case of this class was studied by A.R.S Juma et al.[5].Also the subclass of Multivalent meromorphic harmonic functions with

$$f=z^{p}+\sum_{n=1}^{\infty}\left|a_{n+p-1}\right|z^{-(n+p-1)}-\sum_{n=1}^{\infty}\left|b_{n+p-1}\right|z^{-(n+p-1)},|\,b_{p}\,|<1 \qquad (7)$$

that satisfies(6) is denoted by $\overline{GO_H}(\beta,\lambda,k,p)$ The particular case $\lambda=0$, $p=1$ recently is investigated by [4] for univalent harmonic functions defined in $U=\{z:|\,z\,|<1\}$..
We also note that the class of harmonic multivalent meromorphic functions have been studied by O.P.Ahuja and J.M. Jahangiri [1].
In order to study the various properties of $GO_H(\beta,\lambda,k,p)$) we need the following result Due to O.P. Ahuja and J.M. Jahangiri[1].

Theorem 1: Let $f=h+\overline{g}$ with h and g are given by (2). If

$$\sum_{n=1}^{\infty}(n+p-1)(\left|a_{n+p-1}\right|+|\,b_{n+p-1}\,|\leq p.$$

Then f is harmonic, sense preserving and multivalent in $\overline{U}$ and $f\in\sum_{p}(H)$.

Theorem 2: Let $f=h+\overline{g}$ with h and g are given by (2)**.**Furthermore **,**let

$$\sum_{n=1}^{\infty}\{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]\left|a_{n+p-1}\right|+[2(n+p-1)-p(\alpha+2k)$$
$$\times(\lambda n-\lambda+1)]\,|\,b_{n+p-1}\,|\}T^{\mu}_{p}(n)\leq p(2-2k-\alpha)$$

(9)

Where $0\leq\alpha<2(1-k),0\leq\lambda\leq1,\frac{1}{2}\leq k<1$ and $T^{\mu}_{p}(n)=\dfrac{(\mu+p)_{p+n}(\beta_1)_{n+p}....(\beta_s)_{n+p}}{(\alpha_1)_{n+p}.....(\alpha_q)_{n+p}}$ .Then f is sense preserving multivalent meromorphic harmonic functions with $f\in GO_H(\beta,\lambda,k,p)$

Proof : If the inequality (9) holds for coefficients of $f=h+\overline{g}$, then by (8), f is sense preserving and harmonic multivalent in $\overline{U}$. Now we want to show that $f\in GO_H(\beta,\lambda,k,p)$. According to (6) we have

$$\operatorname{Re}\left\{\frac{[z(H^{\mu}_{p,q,s}h(z))'-\overline{z(H^{\mu}_{p,q,s}g(z))'}](1+e^{i\theta})}{\lambda z(H^{\mu}_{p,q,s}h(z))'-\overline{z(H^{\mu}_{p,q,s}g(z))'})+(1-p\lambda)(H^{\mu}_{p,q,s}h(z)-\overline{H^{\mu}_{p,q,s}g(z)})}-pk(1+e^{i\xi})\right\}\geq p\alpha\,,$$

where $z=re^{i\gamma},0\leq\gamma\leq2\pi,0\leq r<1,0\leq\alpha<2(1-k),0\leq\lambda\leq1,\frac{1}{2}\leq k<1,0\leq\theta<1,0\leq\xi<1.$

Let $N(\lambda,z)=z(H^{\mu}_{p,q,s}h(z))'-\overline{z(H^{\mu}_{p,q,s}g(z))'}](1+e^{i\theta})-pk(1+e^{i\xi})\,[$
$$\lambda z(H^{\mu}_{p,q,s}h(z))'-\overline{z(H^{\mu}_{p,q,s}g(z))'})+(1-p\lambda)(H^{\mu}_{p,q,s}h(z)-\overline{H^{\mu}_{p,q,s}g(z)})\,]$$

and

$$M(\lambda,z)=\lambda z(H^{\mu}_{p,q,s}h(z))'-\overline{z(H^{\mu}_{p,q,s}g(z))'})+(1-p\lambda)(H^{\mu}_{p,q,s}h(z)-\overline{H^{\mu}_{p,q,s}g(z)}).$$

By using the fact $\operatorname{Re} w > p\alpha$ if and only if $|p(1-\alpha)+w| > |p(1+\alpha)-w|$, it is enough to show that

$$|p(1-\alpha)M(\lambda,z)+N(\lambda,z)|-|N(\lambda,z)-p(1+\alpha)M(\lambda,z)|\geq 0.$$

Therefore

$$\begin{aligned}
&|N(\lambda,z)+p(1-\alpha)M(\lambda,z)| = |(1+e^{i\theta})(z(H^{\mu}_{p,q,s}h(z))'-\overline{z(H^{\mu}_{p,q,s}g(z))'})-pk(1+e^{i\xi})\times\\
&[\lambda(z(H^{\mu}_{p,q,s}h(z))'-\overline{z(H^{\mu}_{p,q,s}g(z))'}+(1-p\lambda)(H^{\mu}_{p,q,s}h(z)+\overline{H^{\mu}_{p,q,s}g(z)})]+p(1-\alpha)[\lambda(z(H^{\mu}_{p,q,s}h(z))'-\\
&+(1-p\lambda)(H^{\mu}_{p,q,s}h(z)+\overline{H^{(}_{p,q,s}z)})]|\\
&=|[(1+e^{i\theta})-p\lambda k(1+e^{i\xi})+p\lambda(1-\alpha)]z(H^{\mu}_{p,q,s}h(z))'\\
&-[(1-p\lambda)pk(1+e^{i\xi})-p(1-\alpha)(1-p\lambda)]H^{\mu}_{p,q,s}h(z)-\overline{[(1+e^{i\theta})-\lambda pk(1+e^{i\xi})+p(1-\alpha)\lambda]z(H^{\mu}_{p,}}\\
&-\overline{[pk((1+e^{i\xi})(1-p\lambda)-p(1-\alpha)(1-p\lambda)]H^{\mu}_{p,q,s}g(z)}|\\
&=|[(1+e^{i\theta})-p\lambda k(1+e^{i\xi})+p\lambda(1-\alpha)][pz^{p}-\sum_{n=1}^{\infty}(n+p-1)a_{n+p-1}T^{\mu}_{p}(n)z^{-(n+p-1)}]\\
&-[(1-p\lambda)pk(1+e^{i\xi})-p(1-\alpha)(1-p\lambda)][z^{p}+\sum_{n=1}^{\infty}a_{n+p-1}T^{\mu}_{p}(n)z^{-(n+p-1)}]\\
&-\overline{[(1+e^{i\theta})-p\lambda k(1+e^{i\xi})+p(1-\alpha)\lambda][-\sum_{n=1}^{\infty}(n+p-1)b_{n+p-1}T^{\mu}_{p}(n)z^{-(n+p-1)}]}\\
&-\overline{[pk(1+e^{i\xi})(1-p\lambda)-p(1-\alpha)(1-p\lambda)]\sum_{n=1}^{\infty}b_{n+p-1}T^{\mu}_{p}(n)z^{-(n+p-1)}}\\
&\geq(3p-2kp-p\alpha)|z|^{p}-\sum_{n=1}[2(n+p-1)+p(1-\alpha-2k)(\lambda n+2p\lambda-\lambda-1)]\times\\
&|a_{n+p-1}|T^{\mu}_{p}(n)|z|^{-(n+p-1)}-\sum_{n=1}^{\infty}[2(n+p-1)+p(1-\alpha-2k)(\lambda n-\lambda+1)]|b_{n+p-1}|T^{\mu}_{p}(n)|z|^{-(n+p-1)}
\end{aligned}$$

Also we have

$$|N(\lambda,z)-p(1+\alpha)M(\lambda,z)| = |(1+e^{i\theta})(z(H^{\mu}_{p,q,s}h(z))' - \overline{z(H^{\mu}_{p,q,s}g(z))'} - pk(1+e^{i\xi}) \times$$
$$[\lambda(z(H^{\mu}_{p,q,s}h(z))' - \overline{z(H^{\mu}_{p,q,s}g(z))'}) + (1-p\lambda)(H^{\mu}_{p,q,s}h(z) + \overline{H^{\mu}_{p,q,s}g(z)})]$$
$$-p(1+\alpha)[\lambda(z(H^{\mu}_{p,q,s}h(z))' - \overline{z(H^{\mu}_{p,q,s}g(z))'}) + (1-p\lambda)(H^{\mu}_{p,q,s}h(z) + \overline{H^{\mu}_{p,q,s}g(z)})]|$$
$$= |[(1+e^{i\theta}) - pk\lambda(1+e^{i\xi}) - p\lambda(1+\alpha)]z(H^{\mu}_{p,q,s}h(z))' - [(1-p\lambda)pk(1+e^{i\xi})$$
$$+p(1+\alpha)(1-p\lambda)]H^{\mu}_{p,q,s}h(z) - \overline{[(1+e^{i\theta}) - \lambda pk(1+e^{i\xi}) - p(1+\alpha)\lambda]z(H^{\mu}_{p,q,s}g(z))'}$$
$$-\overline{[pk(1+e^{i\xi})(1-p\lambda) + p(1+\alpha)(1-p\lambda)]H^{\mu}_{p,q,s}g(z)}|$$
$$\geq (p\alpha + 2pk - p)|z|^p + \sum_{n=1}^{\infty}[2(n+p-1) - p(1+\alpha+2k)(\lambda n + 2\lambda p - \lambda - 1)] \times$$
$$|a_{n-p-1}|T^{\mu}_p(n)|z|^{-(n+p-1)} + \sum_{n=1}^{\infty}[2(n+p-1) - p(2k+1+\alpha)(\lambda n - \lambda + 1)]|b_{n+p-1}|T^{\mu}_p(n)|z|^{-(n+p-1)}.$$

Thus,

$$\left|p(1-\alpha)M(\lambda,z)+N(\lambda,z)\right| - \left|N(\lambda,z)-p(1+\alpha)M(\lambda,z)\right|$$
$$\geq 2p(2-2k-\alpha)|z|^p - 2\sum_{n=1}^{\infty}\ [2(n+p-1)-p(\alpha+2k)(\lambda n+2\lambda p-\lambda-1)]\left|a_{n+p-1}\right|T^{\mu}_p(n)|z|^{-(n+p-1)} -$$
$$-2\sum_{n=1}^{\infty}[2(n+p-1)-p(\alpha+2k)(\lambda n-\lambda+1)]\left|b_{n+p-1}\right|T^{\mu}_p(n)\,|\,z\,|^{-(n+p-1)} \geq 0 \ \text{(by (9))}.$$

So, $f \in GO_H(\beta,\lambda,k,p)$.

Theorem 3: Let $f = h + \overline{g}$ with h and g are given by (2)**.**Then $f \in \overline{GO_H}(\beta,\lambda,k,p)$ if and only if

$$\sum_{n=1}^{\infty}\{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]a_{n+p-1} + [2(n+p-1)-p(\alpha+2k)$$
$$\times(\lambda n-\lambda+1)]b_{n+p-1}\}T^{\mu}_p(n)/\,p(2-2k-\alpha) \leq 1 \tag{10}$$

Proof: The "if" part is clear, since $\overline{GO_H}(\beta,\lambda,k,p) \subset GO_H(\beta,\lambda,k,p)$, for "only if" part we show that $f \notin \overline{GO_H}(\beta,\lambda,k,p)$ if the inequality (10) does not hold. So, we must show that

$$\mathrm{Re}\left\{\left(\frac{z(H^{\mu}_{p,q,s}h(z))' - \overline{z(H^{\mu}_{p,q,s}g(z))'}}{\lambda z(H^{\mu}_{p,q,s}h(z))' - \overline{z(H^{\mu}_{p,q,s}g(z))'}) + (1-p\lambda)(H^{\mu}_{p,q,s}h(z) - \overline{H^{\mu}_{p,q,s}g(z)})}\right)(1+pk(1+e^{i\xi}) - p\alpha)\right\}$$

$$= \mathrm{Re}\left\{\frac{C(z)}{D(z)}\right\} \geq 0, \text{ where}$$

$$C(z) = z(H^{\mu}_{p,q,s}h(z))' - \overline{z(H^{\mu}_{p,q,s}g(z))'}](1+e^{i\theta}) - pk(1+e^{i\xi}) + \alpha) \times$$
$$[\lambda z(H^{\mu}_{p,q,s}h(z))' - \overline{z(H^{\mu}_{p,q,s}g(z))'}) + (1-p\lambda)(H^{\mu}_{p,q,s}h(z) - \overline{H^{\mu}_{p,q,s}g(z)})],$$

also

$$D(z) = \lambda(z(H^{\mu}_{p,q,s}h(z))' - \overline{z(H^{\mu}_{p,q,s}g(z))'}] + (1-p\lambda)\ (H^{\mu}_{p,q,s}h(z) - \overline{H^{\mu}_{p,q,s}g(z)})\ ,$$

then

$$C(z)=p(2-2k-\alpha)\,|\,z\,|^{p}\;-\sum_{n=1}^{\infty}\{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]a_{n+p-1}\,|\,T_{p}^{\mu}(n)\times$$
$$|\,z\,|^{-(n+p-1)}\;+[2(n+p-1)-p(\alpha+2k)(\lambda n-\lambda+1)]b_{n+p-1}\}T_{p}^{\mu}(n)\,|\,z\,|^{-(n+p-1)}$$

and

Upon choosing the values of z on the positive real axis, where $z=r<1$,then we must show that

$$\{(2-2k-\alpha)-\sum_{n=1}^{\infty}\{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]\left|a_{n+p-1}\right|+$$
$$+[2(n+p-1)-p(\alpha+2k)(\lambda n-\lambda+1)]\left|b_{n+p-1}\right|\}T_{p}^{\mu}(n)/\{1-\sum_{n=1}^{\infty}\;(\lambda n+2p\lambda-\lambda-1)\left|a_{n+p-1}\right|T_{p}^{\mu}(n)r^{-(n+2p-1)}$$

$$+\sum_{n=1}^{\infty}\;(\lambda n-\lambda+1)\left|b_{n+p-1}T_{p}^{\mu}(n)\right|\,|\,r\,|^{-(n+2p-1)}\}\geq 0. \qquad (11)$$

We note that the last inequality is negative for r sufficiently to 1 , then the inequality (10) does not hold, therefore $\operatorname{Re}\left\{\frac{C(z)}{D(z)}\right\}$ is negative .This contradicts the required condition for $f\in\overline{GO_{H}}(\beta,\lambda,k,p)$.

Next we obtain the distortion bounds and extreme points.

Theorem 4: (Distortion Bounds) : Let $f\in\overline{GO_{H}}(\beta,\lambda,k,p)$ .Then

$$\left|f(z)\right|\leq r^{p}+p(2-2k-\alpha)r^{-p}\;\text{ and }\;\left|f(z)\right|\geq r^{p}-p(2-2k-\alpha)r^{-p}$$

Proof: Let $f\in\overline{GO_{H}}(\beta,\lambda,k,p)$ .Then for $\left|z\right|=r>1$, we have

$$\left|f(z)\right|=|\,z^{p}+\sum a_{n+p-1}T_{p}^{\mu}(n)\,|\,z\,|^{-(n+p-1)}\;-\sum b_{n+p-1}T_{p}^{\mu}(n)\overline{z}^{-(n+p-1)}\;|$$
$$\leq r^{p}+\sum_{n=1}^{\infty}(a_{n+p-1}+b_{n+p-1})T_{p}^{\mu}(n)r^{-(n+p-1)}$$
$$\leq r^{p}+r^{-p}\sum_{n=1}^{\infty}(\left|a_{n+p-1}\right|+\left|b_{n+p-1}\right|)T_{p}^{\mu}(n)r^{-(n+2p-1)}$$
$$\leq r^{p}+r^{-p}\sum_{n=1}^{\infty}\{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]\left|a_{n+p-1}T_{p}^{\mu}(n)\right|r^{-(n+2p-1)}$$
$$+[2(n+p-1)-p(\alpha+2k)(\lambda n-\lambda+1)]\left|b_{n+p-1}T_{p}^{\mu}(n)\right|r^{-(n+2p-1)}]\}$$
$$\leq r^{p}+p(2-2k-\alpha)r^{-p}.$$

The left hand inequality can be proved by using similar arguments.

Theorem 5: (Extreme point): The function $f(z)=h(z)+\overline{g(z)}\;\in\overline{GO_{H}}(\beta,\lambda,k,p)$ if and only if $\;f(z)=\sum(S_{n+p-1}h_{n+p-1}(z)+T_{n+p-1}g_{n+p-1}(z)),z\in\overline{U}\,,\,p\geq 1 \qquad (12)$

Where

$$h_{p-1}(z)=z^{p}, h_{n+p-1}(z)=z^{p}+\frac{p(2-2k-\alpha)}{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]T_{p}^{\mu}(n)}z^{-(n+p-1)},$$

$$g_{p-1}(z)=z^{p}, g_{n+p-1}(z)=z^{p}-\frac{p(2-2k-\alpha)}{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]T_{p}^{\mu}(n)}z^{-(n+p-1)}$$

For $(n\geq 1), \sum_{n=0}^{\infty}(S_{n+p-1}+T_{n+p-1})=1, S_{n+p-1}\geq 0, T_{n+p-1}\geq 0$ .In particular the extreme points of $\overline{GO_{H}}(\beta,\lambda,k,p)$ are $\{h_{n+p-1}\}$ and $\{g_{n+p-1}\}$.

Proof: Suppose that f can be written of the form (12),then

$$\begin{aligned} f(z)&=S_{p-1}h_{p-1}(z)+T_{p-1}g_{p-1}(z)\\ &+\sum_{n=1}^{\infty}S_{n+p-1}(z^{p}+\frac{p(2-2k-\alpha)}{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]T_{p}^{\mu}(n)}z^{-(n+p-1)})\\ &+\sum_{n=1}^{\infty}T_{n+p-1}(z^{p}-\frac{p(2-2k-\alpha)}{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]T_{p}^{\mu}(n)}\overline{z}^{-(n+p-1)})\end{aligned}$$

$$=\sum_{n=0}^{\infty}(S_{n+p-1}+T_{n+p-1})z^{p}+\sum_{n=1}^{\infty}[\frac{p(2-2k-\alpha)}{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]T_{p}^{\mu}(n)}S_{n+p-1}z^{-(n+p-1)}-$$
$$\frac{p(2-2k-\alpha)}{[2(n+p-1)-p(\alpha+2k)(\lambda n-\lambda+1)]T_{p}^{\mu}(n)}T_{n+p-1}\overline{z}^{-(n+p-1)}].$$

Since

$$\sum_{n=1}^{\infty}\{T_{p}^{\mu}(n)(2(n+p-1)-p(\alpha+2k)(\lambda n+2\lambda p-\lambda-1))\times$$

$$\frac{p(2-2k-\alpha)}{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]T_{p}^{\mu}(n)}S_{n+p-1})+(2(n+p-1)-p(\alpha+2k)(\lambda n-\lambda+1))\times$$

$$\frac{p(2-2k-\alpha)}{[2(n+p-1)-p(\alpha+2k)(\lambda n-\lambda+1)]T_{p}^{\mu}(n)}T_{n+p-1})\}=p(2-2k-\alpha)\sum(S_{n+p-1}+T_{n+p-1})$$
$$\leq p(2-2k-\alpha).$$

So by Theorem 3, $f\in\overline{GO_{H}}(\beta,\lambda,k,p)$.

Conversely, if $f\in\overline{GO_{H}}(\beta,\lambda,k,p)$, then by Theorem 3 we have

$$\sum_{n=1}^{\infty}\frac{1}{p(2-2k-\alpha)}[(2(n+p-1)-p(\alpha+2k)(\lambda n+2\lambda p-\lambda-1))a_{n+p-1}T_{p}^{\mu}(n)+$$
$$(2(n+p-1)-p(\alpha+2k)(\lambda n-\lambda+1))b_{n+p-1}T_{p}^{\mu}(n)]\leq 1.$$

Putting $S_{n+p-1}=\dfrac{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]T_p^{\mu}(n)}{p(2-2k-\alpha)}$

$T_{n+p-1}=\dfrac{[2(n+p-1)-p(\alpha+2k)(\lambda n-\lambda+1)]T_p^{\mu}(n)}{p(2-2k-\alpha)}, 0\le S_{p-1}\le 1$ and

$T_{p-1}=1-S_{p-1}-\sum_{n=1}^{\infty}(S_{n+p-1}+T_{n+p-1})$, we get required result.

We define the convolution of f and g as $f*g$ by the following

$$(f*g)(z)=z^p+\sum_{n=1}^{\infty}a_{n+p-1}c_{n+p-1}z^{-(n+p-1)}-\sum_{n=1}^{\infty}b_{n+p-1}d_{n+p-1}\bar{z}^{-(n+p-1)}$$

for

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{n+p-1}z^{-(n+p-1)}-\sum_{n=1}^{\infty}b_{n+p-1}\bar{z}^{-(n+p-1)}$$

and

$$g(z)=z^p+\sum_{n=1}^{\infty}c_{n+p-1}z^{-(n+p-1)}-\sum_{n=1}^{\infty}d_{n+p-1}\bar{z}^{-(n+p-1)}.$$

Now we verify the convolution and convex combination properties of the class $\overline{GO_H}(\beta,\lambda,k,p)$.

Theorem 6: If $f(z)\in\overline{GO_H}(\alpha,\lambda,k,p)$ and $g(z)\in\overline{GO_H}(\beta,\lambda,k,p)$, then for $0\le\beta\le\alpha<2(1-k), f*g\in\overline{GO_H}(\alpha,\lambda,k,p)\subset\overline{GO_H}(\beta,\lambda,k,p)$.

Proof: By assumption $f(z)$ and $g(z)$ satisfy the coefficient inequality (10), then we have

$$\sum_{n=1}^{\infty}[\frac{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]T_p^{\mu}(n)}{p(2-2k-\alpha)}a_{n+p-1}c_{n+p-1}+$$
$$\sum_{n=1}^{\infty}\frac{[2(n+p-1)-p(\alpha+2k)(\lambda n-\lambda+1)]T_p^{\mu}(n)}{p(2-2k-\alpha)}b_{n+p-1}d_{n+p-1}]$$
$$\le\sum_{n=1}^{\infty}[\frac{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]}{p(2-2k-\alpha)}a_{n+p-1}+$$
$$\sum_{n=1}^{\infty}\frac{[2(n+p-1)-p(\alpha+2k)(\lambda n-\lambda+1)]}{p(2-2k-\alpha)}b_{n+p-1}]T_p^{\mu}(n).$$

The last inequality bounded by 1 , since $f\in\overline{GO_H}(\alpha,\lambda,k,p)$ so ,

$$f*g\in\overline{GO_H}(\alpha,\lambda,k,p)\subset\overline{GO_H}(\beta,\lambda,k,p).$$

Theorem 7 : Te family $\overline{GO_H}(\alpha,\lambda,k,p)$ is closed under convex combination .

Proof : Let $f_j(z)\in\overline{GO_H}(\alpha,\lambda,k,p)$ for $j\ge 1$ be defined by the following form

$$f_j(z)=z^p+\sum_{n=1}^{\infty}a_{n+p-1,j}z^{-(n+p-1)}-\sum_{n=1}^{\infty}b_{n+p-1,j}\bar{z}^{-(n+p-1)},(a_{n+p-1,j}\ge 0 and b_{n+p-1,j}\ge 0).$$

Therefore by Theorem 3 , we have

$$\sum_{n=1}^{\infty}[\frac{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]}{p(2-2k-\alpha)}a_{n+p-1,j}+$$

$$\sum_{n=1}^{\infty}\frac{[2(n+p-1)-p(\alpha+2k)(\lambda n-\lambda+1)]}{p(2-2k-\alpha)}b_{n+p-1,j}]T_p^{\mu}(n)\leq 1. \quad \ldots\ldots\ldots\ldots(13)$$

Then we can write for $\sum_{n=1}^{\infty}S_j=1, 0\leq S_j\leq 1,$ the convex combination of f_j as

$$\sum_{j=1}^{\infty}S_j f_j(z)=z^p+\sum_{n=1}^{\infty}(\sum_{j=1}^{\infty}S_j a_{n+p-1,j})z^{-(n+p-1)}-\sum_{n=1}^{\infty}(\sum_{j=1}^{\infty}S_j b_{n+p-1,j})\overline{z}^{-(n+p-1)}.$$

From (13) we get

$$\sum_{n=1}^{\infty}[\frac{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]T_p^{\mu}(n)}{p(2-2k-\alpha)}(\sum_{j=1}^{\infty}S_j a_{n+p-1,j})+$$

$$\frac{[2(n+p-1)-p(\alpha+2k)(\lambda n-\lambda+1)]T_p^{\mu}(n)}{p(2-2k-\alpha)}(\sum_{j=1}^{\infty}S_j b_{n+p-1,j})]$$

$$=\sum_{j=1}^{\infty}S_j\{\sum_{n=1}^{\infty}[\frac{[2(n+p-1)-p(\alpha+2k)(\lambda n+2p\lambda-\lambda-1)]T_p^{\mu}(n)}{p(2-2k-\alpha)}a_{n+p-1,j}+$$

$$\frac{[2(n+p-1)-p(\alpha+2k)(\lambda n-\lambda+1)]T_p^{\mu}(n)}{p(2-2k-\alpha)}b_{n+p-1,j}]\}\leq\sum_{j=1}^{\infty}S_j=1.$$

Therefore

$$\sum_{j=1}^{\infty}S_j f_j(z)\in\overline{GO_H}(\alpha,\lambda,k,p).$$

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