On Investigation of the convergence by least square method for solving Fredholm integral equation of the second kind

Mohammad Shami Hasso   **Aml Jassim Mohammed**

Deportment of math / College of Education

University of Mosul

Received 11/03/2007   **Accepted** 17/07/2007

الملخص

في هذا البحث تم استخدام طريقة المربعات الصغرى المحسنة لحل معادلة فريدهولم التكاملية من النوع الثاني .هذه الطريقة تجمع طريقة المربعات الصغرى باستخدام المتعددات التقدمية والتراجعية ومتعددة حدود شبه خطية متعامدة وان المتعددات تحسن الطريقة أعلاه وتؤدي بها إلى تقارب قوى مناسب، وكانت النتائج العددية مقبولة مقارنة مع الحل المضبوط .

Abstract

In this paper we used the improved least square method presented to solve Fredholm integral equation of the second kind .This method combines, forward and backward factorial polynomial and quasi orthogonal polynomial. These polynomials improved the above method and getting it to a strong convergence property .The numerical results are agreeable in comparison with exact solution .

1.Introdution

In this paper Fredholm integral equation will be solve by using factorial polynomials and there properties. We focus in our work, on the approximate solution of the second kind with the form :

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)\, dt \qquad (1)$$

where λ is a (possibly complex) scalar parameter, $k(x,y)$, $f(x)$ are known functions (often called the driving term) and $u(x)$ is a solution of the Fredholm integral equation.

In order to consolidate these methods, a well- known techniques called weight residual (Least square method) are applied to determine the parameters associated with these methods [4] . In 1971 Hamming used quasi polynomial to expand the exponential function[3] . In 1981 Watson and Qates used factorial polynomial to express any polynomial in terms of factorial polynomials[6]. Also in 1985, 1989 Froberg and EL-Aloosy expressed any polynomial in the form of factorial polynomial respectively[2]. And Finally in 2005 Talhat was used factorial polynomial and quasi polynomial to express any polynomial[5].

Here a first positive forward , positive backward factorial polynomial and quasi orthogonal polynomial accompanied by least square method to treat the solution of (F.I.E. 2^{nd} k)

The integral appears in this paper computed numerically by using quadratic rules (simpson $\frac{1}{3}$ rule),

<u>2.Basic Definitions</u>

let us consider the following definitions:-

<u>Definition2.1.</u> Orthogonal Functions [5]

The set of functions $\{\alpha_n(t)\},\ n = 0,1,2,....$ are said to be a set of orthogonal functions with respect to the weighting function $w(t)$ on the interval [a,b] if given any function $\alpha_n(t)$ and $\alpha_m(t)$ in this set then

$$\int_a^b w(t)\ \alpha_n(t)\ \alpha_m(t)\ \begin{cases} 0 & if\ n \neq m \\ \phi_k & if\ n = m \end{cases} \qquad (2)$$

<u>Definition 2.2.</u> Orthogonal Polynomial and Least Square Method [1]

Assume $S_n(x)$ approximate solution of F.E.I.E. 2^{nd}.K and let it is of the form

$$S_n(x) = \sum_{j=0}^{n} c_j\ \phi_j\ (x) \qquad (3)$$

where $,\phi_0, \phi_1, ...\phi_n$ form a set of orthogonal polynomial defined on [a,b] and c_j being undetermined constant coefficients.

By substituting (3) in (1) we get

$$S_n(x) = f(x) + \int_a^b k(x,t)\, S_n(t)\, dt + E_n(x, c_0, c_1, ...c_n)$$

(4)

or

$$\sum_{j=0}^{n} c_j\ \phi_j(x) = f(x) + \sum_{j=0}^{n} c_j \int_n^b k(x,t)\ \phi_j(t)\ dt + E_n$$

(5)

where E_n denoted the error involved as a result of assuming the approximate solution $S_n(x)$.

Basically our objective is show to find the $n+1$ coefficients $c_0, c_1, c_2, \ldots c_n$ of the approximate solution (3)

The least square method in short insists on the integral of the square of error on the interval [a,b] being minimum that is

$$\int_a^b E_n^2(x, c_0, c_1, \ldots\ldots c_n)\, w(x)\, dx = \min imum$$

(6)

the necessary conditions for E_n to be minimum are

$$\frac{\partial E_n}{\partial c_j} = 0 \qquad , j = 0,1,\ldots\ldots n \tag{7}$$

3. Factorial polynomials:

3.1 Forward Factorization polynomial [6],[5]:-

The symbol [ ] denotes for difference between ordinary power and factorial power. Where the power positive is defined as:-

$$u^{[0]} = 1$$

$$u^{[1]} = u$$

$$u^{[2]} = u(u-1)$$

$$u^{[3]} = u(u-1)(u-2) \tag{8}$$

.

.

$$u^{[n]} = u(u-1)(u-2)\ldots\ldots(u-n+1)$$

3.2 Backward Factorization polynomial [2],[5] :-

the symbol { } denotes for difference of other power is called backward pseudo. Where the power positive is defined as:

$$u^{\{0\}}=1$$

$$u^{\{1\}}=u$$

$$u^{\{2\}}=u(u+1)$$

$$u^{\{3\}}=u(u+1)(u+2) \tag{9}$$

.

.

$$u^{\{n\}}=u(u+1)(u+2)......(u+n-1)$$

where $u=\dfrac{x-x_0}{h}$, x_0 is initial value $,h=\dfrac{b-a}{n}$

3.3 Quasi orthogonal polynomial [3] :-

A construction of family of quasi orthogonal polynomials by a sequence of polynomials interlacing having zero , this rule in the interval say $-1\le x\le 1$

$$P_0(x)=1$$

$$P_1(x)=x$$

$$P_2(x)=(x-\frac{1}{2})(x+\frac{1}{2})$$

$$P_3(x)=(x-\frac{2}{3})(x+\frac{2}{3})\,x \tag{10}$$

.

.

$$P_n(x)=(x-\frac{(n-1)}{n})(x+\frac{(n-1)}{n})\,x^{n-2} \qquad ,n\ge 2$$

3.4 Using Positive Power Forward Factorial Polynomial to Solve (F.I.E.2nd K.)

In this section the (P.P.F.F.P) accompanied by least square method will be used to find the numerical solution of (F.I.E.2nd K).An approximate solution for integral equation of the form

$$S_n(x)=\sum_{j=o}^{n}c_j u^{[k]} \tag{11}$$

will be assumed substituting $S_n(x)$ in the equation (1) we get the equation

$$\sum_{j=0}^{n} c_j u^{[k]} = f(x) + \sum_{j=0}^{n} c_j \int_a^b k(x,t) u^{[k]} dt + E_n(x)$$
(12)

by using Least square method we obtained the following system

$$\sum_{j=0}^{n} a_{kj} c_j = b_k \qquad ,k = 0,1,2,..,n \tag{13}$$

Here we get a system of $(n+1)$ linear equations with $(n+1)$ unknown coefficients $c_0, c_1, c_2, \ldots\ldots c_n$.

By solving equation (13) we find the value of c_j's and then substitute them in equation (11) the approximate solution of equation (1) is given.

We used the Quadrate rule (sim $\frac{1}{3}$) to solve the required integral numerically.

By using similar steps to find the equation of backward factorial polynomial and quasi orthogonal polynomial to solve (F.I.E.2nd K) respectively

$$\sum_{j=0}^{n} c_j u^{\{k\}} = f(x) + \sum_{j=0}^{n} c_j \int_a^b k(x,t) u^{\{k\}} dt + E_n(x)$$
(14)

$$\sum_{j=0}^{n} c_j P_j(x) = f(x) + \sum_{j=0}^{n} c_j \int_a^b k(x,t) P_j(t) dt + E_n(x) \tag{15}$$

4 Numerical Results:

the numerical solution of Fredholm integral equation is introduced using several numerical methods. Examples are solved and good a results are achieved.

A comparison is made between these methods depending on least square error (L.S.E), which is calculated from the numerical solution against the exact solution .

As a first example we consider the equation

$$u(x) = f(x) + \int_0^1 \exp(tx) u(t) dt$$

with $f(x) = \exp(x) - [\exp(x+1) - 1]/(x+1)$, which has the exact solution $u(x) = \exp(x)$.

In table 1 we give some result obtained for this specified problem by using L.S.M.F , L.S.M.B and L.S.M.Q.

Table 1 comparison between the exact solution and the numerical solution

X	Exact	L.S.M.F	L.S.M.B	L.S.M.Q
0.00	1.0000000	0.9997801	0.9997815	0.99978015
0.17	1.18136041	1.18112872	1.8112872	1.1811287
0.33	1.3956124	1.395375	1.3953722	1.395372
0.50	1.6487212	1.648478	1.6484785	1.6484785
0.67	1.9477340	1.947498	1.947498	1.9474985
0.83	2.3009758	2.3007628	2.3007628	2.3007628
1.00	2.7182818	2.7181135	2.71811353	2.7181135
L.S.E		3.477795-7	3.477797-7	3.477795-7

A second example illustrates in an optimal way the practical use forward , backward and quasi formula

The example is

$$u(x) = f(x) + \int_0^{\pi} k(x,t)\ u(t)\,dt$$

with $f(x) = \sin(x) - x\pi$, $k(x,t) = xt$

which has the exact solution

$u(x) = \sin(x)$ *

In table (2) we give some results obtained for the specific problem. It is clear that the data in the extrapolation approach reproduce the approximate solution *

Table (2) comparison between the exact solution and the numerical solution

X	Exact	L.S.M.F	L.S.M.B	L.S.M.Q
0.00	0.000000	0.00000025	0.0000012	0.00000940
0.17	0.500000	0.4999239	0.4999239	0.4999239
0.33	0.866025	0.865973	0.8658733	0.8658733
0.50	1.000000	0.9997718	0.9997718	0.9997718
0.67	0.866025	0.8657212	0.8657212	0.8657212
0.83	0.5000000	0.499619757	0.4996197	0.4996197
1.00	1.22464-16	-0.0051562	-0.00045628	-0.000456
L.S.E		5.2629-007	5.2624-007	5.2628-007

Conclusion:

The improved least square method which was used with positive forward , positive backward factorial polynomials and quasi orthogonal polynomial were applied to find the numerical solution of (F.I.E.2^{nk} k) .

The method in this paper has a strong converges and will yield a batter accuracy to a chive the exact solution.

References

[1] Chambers, L.G., (1976), " ***Integral Equations A Short Course*** " , International Texbook.Com.Limited

[2] Froberg , C.E., (1985) , " ***Numerical Mathematics*** " ,the Benjamin / Cummings publishing com.inc. New York.

[3] Hamming , R.W.,(1971), " ***Introduction to Applied Numerical Analysis*** ", Inferential student edition.

[4] Kalwi , P.W , (1999) , " ***Numerical Methods for Solving Fredholm Integral Equations of the Second Kind*** ", M.Sc, Thesis Universe of Technology.

[5] Tahlat , I.H ,(2005) ," ***Numerical Methods for Solving Fredholm Integral Equation of the First Kind with Degenerate Kernel*** ", M.Sc,Thesis Universe of Salahadden –Erbil.

[6] Watson , W.A . and Qates , P.J. , (1981) , " ***Numerical Analysis the Mathematics of Computing*** " second addition .Edward Arnold.