# Existence of a periodic solutions for certain system of nonlinear integro-differential equations

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**الملخص**

يتضمن البحث دراسة وجود وتقريب الحلول الدورية لبعض أنظمة المعادلات التكاملية–التفاضلية غير الخطية وذلك باستخدام طريقة التقريب المتتابع للحلول الدورية للمعادلات التفاضلية الاعتيادية لـ A. M. Samoilenko. وكذلك تؤدي هذه الدراسة الى تحسين وتوسيع الطريقة أعلاه.

**ABSTRACT**

In this paper we investigate the existence and approximation of the periodic solutions for certain systems of nonlinear integro-differential equations, by using the method of successive periodic approximation of ordinary differential equations which is given by A. M. Samoilenko. Also these investigation lead us to the improving the extending the above method.

**Introduction**

Consider the following system of integro-differential equation, which has the form:

$$\frac{dx}{dt} = f\left(t, x, \dot{x}, \int_{t-T}^{t} g\big(s, x(s), \dot{x}(s)\big)ds\right), \qquad \ldots\ldots (1)$$

where $x \in D \subseteq R^n$, $D$ is a closed and bounded domain.

The vectors functions $f(t,x,\dot{x},v)$ and $g(t,x,\dot{x})$ are defined on the domain:

$$(t,x,\dot{x},v) \in R^1 \times D \times D_1 \times D_2$$
$$= (-\infty,\infty) \times D \times D_1 \times D_2 \qquad \ldots\ldots (2)$$

which are continuous in $(t,x,\dot{x},v)$ and periodic in t of period T , where $D_1$ and $D_2$ are bounded domains subset of Euclidean space $R^m$.

Let the functions $f(t,x,\dot{x},v)$ and $g(t,x,\dot{x})$ are satisfy the following inequalities:

$$|f(t,x,\dot{x},v)| \le M \quad , \quad |g(t,x,\dot{x})| \le M \quad , \qquad \ldots\ldots (3)$$

$$|f(t,x_1,\dot{x}_1,v_1) - f(t,x_2,\dot{x}_2,v_2)| \le K_1|x_1 - x_2| + K_2|\dot{x}_1 - \dot{x}_2| + K_3|v_1 - v_2| \ , \qquad \ldots\ldots (4)$$

$$\left|g(t,x_1,\dot{x}_1)-g(t,x_2,\dot{x}_2)\right| \le L_1|x_1-x_2|+L_2|\dot{x}_1-\dot{x}_2| , \qquad \dots\dots (5)$$

for all $t\in R^1$ and $x,x_1,x_2\in D$ , $\dot{x},\dot{x}_1,\dot{x}_2\in D_1$ and $v,v_1,v_2\in D_2$ , where $M$ is appositive constant vector and $K_1,K_2,K_3,L_1,L_2$, are $(n\times n)$ constant matrices, $|\,.\,|_\circ = \max_{t\in[0,T]}|\,.\,|$ .

We define the non-empty sets as follows:

$$\left.\begin{aligned} D_f &= D-\frac{T}{2}M \\ D_{1f} &= D_1-2M \\ D_{2f} &= D_2-\left[TM+\frac{T}{2}M\left(L_1T+4L_2\right)\right] \end{aligned}\right] \qquad \dots\dots (6)$$

Furthermore, we suppose that the greatest eigen-value of the matrix $W=\left[\left(K_1+K_3L_1T\right)\frac{T}{2}+2\left(K_2+K_3L_2T\right)\right]$ dose not exceeds unity, i.e.

$$\left|\lambda_j(W)\right|<1 \ , \ \left(j=1,2,...,n\right). \qquad \dots\dots (7)$$

**Lemma 1 :**

Let $f(t)$ be a continuous vector function defined in the interval $[0,T]$, then:

$$\left|\int_0^t\left(f(t)-\frac{1}{T}\int_0^T f(s)ds\right)ds\right| \le \alpha(t)\left|f(t)\right|_\circ ,$$

where $\alpha(t)=2t\left(1-\frac{t}{T}\right)$ and $\left|f(t)\right|_\circ=\max_{t\in[0,T]}\left|f(t)\right|$.

For the proof see [ 4 ] .

**Approximation Solution of (1)**

The investigation of periodic approximation solution of the system (1) makes essential use of the statements given below.

**Theorem 1:**

If the system of integro-differential equations (1) satisfy the inequalities (3), (4) with assumptions (5) and the conditions (6), (7) has a periodic solution $x=x(t,x_0)$, passing through the point $(0,x_0)$, $x_0\in D_f$, then the sequence of functions:

$$x_{m+1}(t,x_0)=x_0+\int_0^t\left[f\left(s,x_m(s,x_0),\dot{x}_m(s,x_0),\int_{s-T}^{s}g\left(\tau,x_m(\tau,x_0),\dot{x}_m(\tau,x_0)\right)d\tau\right)-\right.$$

$$-\frac{1}{T}\int_0^T f\left(s, x_m(s,x_0), \dot{x}_m(s,x_0), \int_{s-T}^{s} g\left(\tau, x_m(\tau,x_0), \dot{x}_m(\tau,x_0)\right)d\tau\right)\Bigg]$$

… … (8)

with

$$x_0(t,x_0)=x_0 \quad , \quad \frac{dx_m(t,x_0)}{dt}=\dot{x}_m(t,x_0) \quad , \quad m=0,1,2,\ldots$$

is periodic in t of period T, and then is uniformly convergent as $m\to\infty$ in the domain:

$$(t,x_0)\in R^1\times D_f = (-\infty,\infty)\times D_f \quad , \qquad \text{… … (9)}$$

to the function $x_\infty(t,x_0)$ defined in the domain (9), which is periodic in t of period T and satisfying the system of equations:

$$x(t,x_0)=x_0+\int_0^t\Bigg[f\left(s, x(s,x_0), \dot{x}(s,x_0), \int_{s-T}^{s} g\left(\tau, x(\tau,x_0), \dot{x}(\tau,x_0)\right)d\tau\right)-$$
$$-\frac{1}{T}\int_0^T f\left(s, x(s,x_0), \dot{x}(s,x_0), \int_{s-T}^{s} g\left(\tau, x(\tau,x_0), \dot{x}(\tau,x_0)\right)d\tau\right)ds\Bigg]dt$$

… … (10)

which is a unique solution of the system (1).

**Proof:**

Consider the sequence of functions $x_1(t,x_0), x_2(t,x_0),\ldots, x_m(t,x_0),\ldots$, defined by recurrence relation (8). Each of the functions of the sequence are periodic in t of period T.

Now, by the lemma 1, we have from (8), for m=0:

$$|x_1(t,x_0)-x_0|\le\left(1-\frac{t}{T}\right)\int_0^t\left|f\left(s, x_0, 0, \int_{s-T}^{s} g\left(\tau, x_0, 0\right)d\tau\right)\right|ds+$$
$$+\frac{t}{T}\int_t^T\left|f\left(s, x_0, 0, \int_{s-T}^{s} g\left(\tau, x_0, 0\right)d\tau\right)\right|ds$$
$$\le M\alpha(t)\le M\frac{T^2}{6} \qquad \text{… … (11)}$$

It follows that $x_1(t,x_0)\in D$ for all $t\in R^1$ and $x_0\in D_f$. Moreover, on differentiating $x_1(t,x_0)$, we find

$$\dot{x}_1(t,x_0)=f\left(t, x_0, 0, \int_{t-T}^{t} g\left(s, x_0, 0\right)ds\right)-\frac{1}{T}\int_0^T f\left(t, x_0, 0, \int_{t-T}^{t} g\left(s, x_0, 0\right)ds\right)dt$$

and hence

$$|\dot{x}_1(t,x_0)| \le \left| f\left(t,x_0,0,\int_{t-T}^{t} g(s,x_0,0)ds\right)\right| + \frac{1}{T}\int_0^T \left| f\left(t,x_0,0,\int_{t-T}^{t} g(s,x_0,0)ds\right)\right| dt$$
$$\le 2M \qquad \dots\dots (12)$$

By condition (6), it follows from the last inequality that $\dot{x}_1(t,x_0) \in D_1$ for all $t \in R^1$ and $x_0 \in D_f$.

Thus by induction we can prove that $x_m(t,x_0) \in D$ and $\dot{x}_m(t,x_0) \in D_1$, for all $m \ge 1$ and $x_0 \in D_f$.

We have to prove that the sequence of functions (8) is uniformly convergent on the domain (9).

By using (8) and (11) the following inequalities are holds:

$$|x_{m+1}(t,x_0) - x_m(t,x_0)| \le \alpha(t) M W^m \qquad \dots\dots (13)$$

and

$$|\dot{x}_{m+1}(t,x_0) - \dot{x}_m(t,x_0)| \le 2MW^m \qquad \dots\dots (14)$$

From (13) and (14) we conclude that for any $k \ge 1$, we have the inequalities:

$$|x_{m+k}(t,x_0) - x_m(t,x_0)| \le \alpha(t) M W^m \sum_{i=0}^{k-1} W^i \quad , \qquad \dots\dots (15)$$

and

$$|\dot{x}_{m+k}(t,x_0) - \dot{x}_m(t,x_0)| \le 2MW^m \sum_{i=0}^{k-1} W^i \qquad \dots\dots (16)$$

It follows from (15) and (16) that:

$$|x_{m+k}(t,x_0) - x_m(t,x_0)| \le \alpha(t) W^m (E-W)^{-1} M \qquad \dots\dots (17)$$

and

$$|\dot{x}_{m+k}(t,x_0) - \dot{x}_m(t,x_0)| \le 2W^m (E-W)^{-1} M \quad , \qquad \dots\dots (18)$$

for all $t \in R^1$ and $k \ge 1$, where $E$ is identity matrix.

From (17), (18) and the condition (7), the sequence of functions $\{x_m(t,x_0), \dot{x}_m(t,x_0)\}$ is uniformly convergent in (9) as $m \to \infty$.

Let

$$\lim_{m\to\infty} x_m(t,x_0) = x_\infty(t,x_0) \qquad \dots\dots (19)$$

and

$$\lim_{m\to\infty} \dot{x}_m(t,x_0) = \dot{x}_\infty(t,x_0) \qquad \dots\dots (20)$$

Since the sequence of functions $x_m(t,x_0)$ and $\dot{x}_m(t,x_0)$ are periodic in t of period T, then the limiting functions $x_\infty(t,x_0) = x(t,x_0)$ and $\dot{x}_\infty(t,x_0) = \dot{x}(t,x_0)$ are periodic in t of period T.

Moreover, by lemma 1 and (17), (18), the following inequalities are holds:

$$\left|x_{m+k}(t,x_0)-x_m(t,x_0)\right|\le \alpha(t)W^{m+1}\left(E-W\right)^{-1}M \qquad \dots\dots (21)$$

and

$$\left|\dot{x}_{m+k}(t,x_0)-\dot{x}_m(t,x_0)\right|\le 2W^{m+1}\left(E-W\right)^{-1}M \qquad \dots\dots (22)$$

for all $m\ge 0$ and $t\in R^1$.

Finally, we have to show that $x(t,x_0)$ is unique solution of the system (1). On the contrary, we suppose that there is at least two different solutions $x(t,x_0)$ and $y(t,x_0)$ of (1).

From (10), the following identity holds:

$$\left|x(t,x_0)-y(t,x_0)\right|_\circ \le \left(K_1+K_3L_1T\right)\frac{T}{2}\left|x(t,x_0)-y(t,x_0)\right|_\circ +$$
$$+\left(K_2+K_3L_2T\right)\frac{T}{2}\left|\dot{x}(t,x_0)-\dot{y}(t,x_0)\right|_\circ \qquad \dots\dots (23)$$

On differentiating (23) we should also obtain

$$\left|\dot{x}(t,x_0)-\dot{y}(t,x_0)\right|_\circ \le 2\left(K_1+K_3L_1T\right)\left|x(t,x_0)-y(t,x_0)\right|_\circ +$$
$$+2\left(K_2+K_3L_2T\right)\left|\dot{x}(t,x_0)-\dot{y}(t,x_0)\right|_\circ \qquad \dots\dots (24)$$

Inequalities (23) and (24) would lead to the estimate

$$\left|x(t,x_0)-y(t,x_0)\right|_\circ \le QW \ , \qquad \dots\dots (25)$$

where

$$Q=T\left[\left(K_1+K_3L_1T\right)\left|x(t,x_0)-y(t,x_0)\right|_0+\left(K_2+K_3L_2T\right)\left|\dot{x}(t,x_0)-\dot{y}(t,x_0)\right|_0\right]$$

and $W=\left[\left(K_1+K_3L_1T\right)\frac{T}{2}+2\left(K_2+K_3L_2T\right)\right]$

By iteration we have

$$\left|x(t,x_0)-y(t,x_0)\right|_\circ \le QW^m \ , \qquad \dots\dots (26)$$

But $W^m\to 0$ as $m\to\infty$, hence proceeding in the last inequality to the limit we obtain that $x(t,x_0)=y(t,x_0)$ which proves that the solution is unique, and this completes the proof of theorem 1.

**Existence of Solution of (1)**

The problem of existence of periodic solution of period T of the system (1) is uniquely connected with the existence of zeros of the function $\Delta(x_0)$, which has the form:-

$$\Delta(x_0)=\frac{1}{T}\int_0^T f\left(t,x_\infty(t,x_0),\dot{x}_\infty(t,x_0),\int_{t-T}^{t} g\left(s,x_\infty(s,x_0),\dot{x}_\infty(s,x_0)\right)ds\right)dt \qquad \dots\dots (27)$$

where $x_\infty(t,x_0)$ is the limit function of the sequence functions $x_m(t,x_0)$.

Since this function is approximately determined from the sequence of functions:

$$\Delta_m(x_0)=\frac{1}{T}\int_0^T f\left(t, x_m(t,x_0), \dot{x}_m(t,x_0), \int_{t-T}^{t} g(s, x_m(s,x_0), \dot{x}_m(s,x_0))ds\right)dt$$

… … (28)

m=0,1,2,… .

Now we prove the following theorems taking into account that the following inequality will be satisfied for all $m \geq 1$.

$$\left|\Delta(x_0)-\Delta_m(x_0)\right|_\circ = \frac{1}{T}\int_0^T \Big[K_1\left|x_\infty(t,x_0)-x_m(t,x_0)\right|_\circ + K_2\left|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\right|_\circ +$$

$$+K_3T\left(L_1\left|x_\infty(t,x_0)-x_m(t,x_0)\right|_\circ + L_2\left|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\right|_\circ\right)\Big]dt$$

$$\leq \frac{1}{T}\int_0^T (K_1+K_3L_1T)\left|x_\infty(t,x_0)-x_m(t,x_0)\right|_\circ dt +$$

$$+\frac{1}{T}\int_0^T (K_2+K_3L_2T)\left|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\right|_\circ dt$$

$$\leq \left[(K_1+K_3L_1T)\frac{T}{2}+2(K_2+K_3L_2T)\right]W^m(E-W)^{-1}M$$

$$=W^{m+1}(E-W)^{-1}M \qquad \text{… … (29)}$$

**Theorem 2:**

If the system of equation (1) satisfies the following conditions:

(**i**) The sequence of functions (28) has an isolated singular point $x_0 = x_\infty$, $\Delta_m(x_\infty)=0$.

(**ii**) The index of this point is nonzero.

(**iii**) There exist a closed convex domain $D_3$ belonging to the domain $D_f$ and possessing a unique singular point $x_\infty$, such that on it's boundary $\Gamma_{D_3}$ the following inequality holds:

$$\inf_{x\in\Gamma_{D_3}} \left|\Delta_m(x)\right| \geq W^{m+1}(E-W)^{-1}M,$$

where $W=\left[(K_1+K_3L_1T)\frac{T}{2}+2(K_2+K_3L_2T)\right]$ and $m\geq 1$. Then the system (1) has a periodic solution $x=x(t)$ for $x(0)\in D_3$.

**Proof:**

By using the inequality (29) we can prove the theorem in a similar way to the theorem 1 [ 3 ].

**Remark 1:** [ 1 ]

When $R^n = R^1$, i.e. when $x$ is a scalar, theorem 2 can be strengthens by giving up the requirement that the singular point should be isolated, thus we have.

**Theorem 3:**

Let the functions $f(t,x,\dot{x},v)$ and $g(t,x,\dot{x})$ of the system (1) are defined on the interval $[a,b]$ in $R^1$. Assume that for any integer $m \geq 1$, the function $\Delta_m(x_0)$ defined according to formula (28) satisfies the inequalities:

$$\left.\begin{aligned} \min_{a+\frac{MT}{2}\leq x\leq b-\frac{MT}{2}} \Delta_m(x_0) &\leq -q^{m+1}(1-q)^{-1}\frac{MT}{2} , \\ \max_{a+\frac{MT}{2}\leq x\leq b-\frac{MT}{2}} \Delta_m(x_0) &\geq q^{m+1}(1-q)^{-1}\frac{MT}{2} , \end{aligned}\right] \qquad \ldots\ldots (30)$$

where $q = (K_1 + K_3L_1T)\frac{T}{2} + 2(K_2 + K_3L_2T)$, and $K_1, K_2, K_3,\ L_1, L_2$ are positive constants. Then the system (1) has a periodic solution of period T, $x = x(t)$ for $x(0) \in \left[a + \frac{MT}{2}, b - \frac{MT}{2}\right]$.

**Proof:**

Let $x_1$ and $x_2$ be any two points of the interval $\left[a + \frac{MT}{2}, b - \frac{MT}{2}\right]$ such that:

$$\left.\begin{aligned} \Delta_m(x_1) &= \min_{a+\frac{MT}{2}\leq x\leq b-\frac{MT}{2}} \Delta_m(x) , \\ \Delta_m(x_2) &= \max_{a+\frac{MT}{2}\leq x\leq b-\frac{MT}{2}} \Delta_m(x) . \end{aligned}\right] \qquad \ldots\ldots (31)$$

By using the inequalities (29) and (30), we have:

$$\left.\begin{aligned} \Delta(x_1) &= \Delta_m(x_1) + (\Delta(x_1) - \Delta_m(x_1)) < 0 , \\ \Delta(x_2) &= \Delta_m(x_2) + (\Delta(x_2) - \Delta_m(x_2)) > 0 . \end{aligned}\right] \qquad \ldots\ldots (32)$$

From the continuity of $\Delta(x_0)$ and (31), (32), there exists a point $x_\infty$; $x_\infty \in [x_1, x_2]$, such that $\Delta(x_\infty) = 0$, and this proves the theorem.

Similar results can be obtained for other class of integro-differential equation. In particular, the system of equations which has the form:

$$\frac{dx}{dt} = f\left(t, x, \dot{x}, \int_{a(t)}^{b(t)} g(s, x(s), \dot{x}(s))ds\right), \qquad \ldots\ldots (33)$$

In this system (33), let the vector functions $f(t,x,\dot{x},w)$, $g(t,x,\dot{x})$ and the scalar functions $a(t)$, $b(t)$ are periodic in t of period T, defined and continuous in $R^1 \times D \times D_1 \times D_2$, $R^1 \times D \times D_1$ and $R^1$, $R^1$. Suppose that the functions $f(t,x,\dot{x},w)$ and $g(t,x,\dot{x})$ satisfying the inequalities (3), (4) and (5) and the conditions (6), with $D_2 f = D_f - \left[ hM \frac{T}{2}(L_1 + 4L_2) + MT \right]$.

Furthermore, the largest eigenvalue $\lambda_{\max}$ of the matrix $\Lambda = \left[ (K_1 + K_3 L_1 h)\frac{T}{2} + 2(K_2 + K_3 L_2 h) \right]$ is less than unity i.e.,

$$\lambda_{\max}(\Lambda) < 1 , \qquad \dots \dots (34)$$

where $h = \max_{t[0,T]} |b(t) - a(t)|$

**Theorem 4:**

If the system of equations (33) satisfies the above assumptions and conditions has a periodic solution $x = \psi(t)$, passing through the point $(0, x_0)$, $x_0 \in D_f$, then their exist a unique solution which is the limit function of a uniformly convergent sequence which has the form:

$$x_{m+1}(t,x_0) = x_0 + \int_0^t \left[ f\left( s, x_m(s,x_0), \dot{x}_m(s,x_0), \int_{a(s)}^{b(s)} g(\tau, x_m(\tau,x_0), \dot{x}_m(\tau,x_0)) d\tau \right) - \right.$$
$$\left. - \frac{1}{T}\int_0^T f\left( s, x_m(s,x_0), \dot{x}_m(s,x_0), \int_{a(s)}^{b(s)} g(\tau, x_m(\tau,x_0), \dot{x}_m(\tau,x_0)) d\tau \right) \right] \qquad \dots \dots (35)$$

with

$x_0(t,x_0) = x_0$ , m=0,1,2,…

The proof is similar to the theorem 1 [1].

If we consider the following sequence of functions:

$$\Delta_m(x_0) = \frac{1}{T}\int_0^T f\left( t, x_m(t,x_0), \dot{x}_m(t,x_0), \int_{a(t)}^{b(t)} g(s, x_m(s,x_0), \dot{x}_m(s,x_0)) ds \right) dt , \qquad \dots \dots (36)$$

then we can state a theorem similar to the theorem 2 provided that: $\lambda_{\max}(\Lambda) < 1$.

Also we can consider the following system of integro-differential equation, which has the form:

$$\frac{dx}{dt} = f\left( t, x, \dot{x}, \int_{-\infty}^{t} G(t,s) g(s, x(s), \dot{x}(s)) ds \right) , \qquad \dots \dots (37)$$

The vectors functions $f(t,x,\dot{x},z)$ and $g(t,x,\dot{x})$ are defined on the domain:

$$\Omega = R' \times D \times D_1 \times D_2 \qquad \ldots\ldots (38)$$

which are continuous in $(t,x,\dot{x},z)$ and periodic in t with period T , where $D_1$ and $D_2$ are bounded domains subsets of Euclidean spaces $R^m$ .

Let the functions $f(t,x,\dot{x},z)$ and $g(t,x,\dot{x})$ are satisfy the following inequalities:

$$|f(t,x,\dot{x},z)| \le M \quad , \quad |g(t,x,\dot{x})| \le M \quad , \qquad \ldots\ldots (39)$$

$$|f(t,x_1,\dot{x}_1,z_1) - f(t,x_2,\dot{x}_2,z_2)| \le K_1|x_1 - x_2| + K_2|\dot{x}_1 - \dot{x}_2| + K_3|z_1 - z_2| \ , \qquad \ldots\ldots (40)$$

$$|g(t,x_1,\dot{x}_1) - g(t,x_2,\dot{x}_2)| \le L_1|x_1 - x_2| + L_2|\dot{x}_1 - \dot{x}_2| \ , \qquad \ldots\ldots (41)$$

for all $t \in R^1$ and $x, x_1, x_2 \in D$ , $\dot{x}, \dot{x}_1, \dot{x}_2 \in D_1$ and $z, z_1, z_2 \in D_2$ , where $M = (M_1, M_2, ..., M_n)$ is a positive constant vector and $K_1, K_2, K_3$, and $L_1, L_2$ are $(n \times n)$ constant matrices. A matrix $G(t,s)$ is defined and continuous in $R^1 \times R^1$ and satisfies the condition $G(t,s) = G(t+T, s+T)$, with $\|G(t,s)\| \le \delta e^{-\gamma(t-s)}$, $0 \le s \le t \le T$, where $\delta, \gamma$ are positive constants, $|\,.\,|_\circ = \max\limits_{0 \le t \le T} |\,.\,|$ , $\|\,.\,\| = \max\limits_{t \in [0,T]} |\,.\,|$ .

We define the non-empty sets as follows:

$$\left.\begin{aligned} D_f &= D - \frac{T}{2}M \\ D_{1f} &= D_1 - 2M \\ D_{2f} &= D_2 - \left[\frac{MT}{2}\frac{\gamma}{\delta}(L_1 + 4L_2) + \frac{M\gamma}{\delta}\right] \end{aligned}\right] \qquad \ldots\ldots (42)$$

Furthermore, we suppose that the greatest eigen-value of the matrix $\Lambda = \left[\left(K_1 + K_3 L_1 \frac{\gamma}{\delta}\right)\frac{T}{2} + 2\left(K_2 + K_3 L_2 \frac{\gamma}{\delta}\right)\right]$ dose not exceeds unity, i.e.

$$|\lambda_j(\Lambda)| < 1 \ , \ (j = 1,2,...,n). \qquad \ldots\ldots (43)$$

**Approximation Solution of (37)**

The investigation of periodic approximation solution of the system (37) will be introduced by the following theorem.

**Theorem 5:**

Let $f \in C(\Omega)$ and satisfies the inequalities (39), (40), (41) with assumptions (42) and the condition (43) are given. Then the sequence of functions $\{x_m(t,x_0)\}$ defined by:

$$x_{m+1}(t,x_0)=x_0+\int_0^t\left[f\left(s,x_m(s,x_0),\dot{x}_m(s,x_0),\int_{-\infty}^{s}G(s,\tau)g\left(\tau,x_m(\tau,x_0),\dot{x}_m(\tau,x_0)\right)d\tau\right)-\right.$$
$$\left.-\frac{1}{T}\int_0^T f\left(s,x_m(s,x_0),\dot{x}_m(s,x_0),\int_{-\infty}^{s}G(s,\tau)g\left(\tau,x_m(\tau,x_0),\dot{x}_m(\tau,x_0)\right)d\tau\right)ds\right]ds$$

… … (44)

with

$$x_0(t,x_0)=x_0 \quad , \quad \frac{dx_m(t,x_0)}{dt}=\dot{x}_m(t,x_0) \quad , \quad m=0,1,2,\ldots$$

convergent uniformly in $[0,T]\times D_f$ to the function $x^{\circ}(t,x_0)$ whish is:

$$x(t,x_0)=x_0+\int_0^t\left[f\left(s,x(s,x_0),\dot{x}(s,x_0),\int_{-\infty}^{s}G(s,\tau)g\left(\tau,x(\tau,x_0),\dot{x}(\tau,x_0)\right)d\tau\right)-\right.$$
$$\left.-\frac{1}{T}\int_0^T f\left(s,x(s,x_0),\dot{x}(s,x_0),\int_{-\infty}^{s}G(s,\tau)g\left(\tau,x(\tau,x_0),\dot{x}(\tau,x_0)\right)d\tau\right)ds\right]ds$$

… … (45)

provided that

$$\left|x^0(t,x_0)-x_m(t,x_0)\right|_{\circ}\le\Lambda^m\left(E-\Lambda\right)^{-1}\frac{MT}{2} \qquad \text{… … (46)}$$

and

$$\left|\dot{x}^0(t,x_0)-\dot{x}_m(t,x_0)\right|_{\circ}\le 2\Lambda^m\left(E-\Lambda\right)^{-1}M \qquad \text{… … (47)}$$

for all $m\ge 1$ and $t\in R^1$.

**Proof:**

Setting m=0 and using (44), we get:

$$\left|x_1(t,x_0)-x_0\right|\le\left(1-\frac{t}{T}\right)\int_0^t\left|f\left(s,x_0,0,\int_{-\infty}^{s}G(s,\tau)g\left(\tau,x_0,0\right)d\tau\right)\right|ds+$$
$$+\frac{t}{T}\int_t^T\left|f\left(s,x_0,0,\int_{-\infty}^{s}G(s,\tau)g\left(\tau,x_0,0\right)d\tau\right)\right|ds$$

So that

$$\left|x_1(t,x_0)-x_0\right|\le M\alpha(t) \qquad \text{… … (48)}$$

So that $x_1(t,x_0)\in D$ for all $t\in R^1$ and $x_0\in D_f$. Moreover, on differentiating $x_1(t,x_0)$, we find

$$\dot{x}_1(t,x_0)=f\left(t,x_0,0,\int_{-\infty}^{t}G(t,s)g\left(s,x_0,0\right)ds\right)-\frac{1}{T}\int_0^T f\left(t,x_0,0,\int_{-\infty}^{t}G(t,s)g\left(s,x_0,0\right)ds\right)dt$$

and hence

$$|\dot{x}_1(t,x_0)| \le \left| f\left(t,x_0,0,\int_{-\infty}^{t} G(t,s)g(s,x_0,0)ds\right)\right| + \frac{1}{T}\int_0^T \left| f\left(t,x_0,0,\int_{-\infty}^{t} G(t,s)g(s,x_0,0)ds\right)\right| dt$$

$$\le 2M \qquad \ldots \ldots (49)$$

From (49) and (42), we get $\dot{x}_1(t,x_0) \in D_1$ for all $t \in R^1$ and $x_0 \in D_f$. Thus by induction we can prove that $x_m(t,x_0) \in D$ and $\dot{x}_m(t,x_0) \in D_1$, for all $t \in R^1$, $x_0 \in D_f$ and m=1,2,… .

We claim that the sequence of functions $x_m(t,x_0)$ is uniformly convergent on the domain $R^1 \times D_f$.

By using (44) and (48) the following inequalities are holds:

$$|x_{m+1}(t,x_0) - x_m(t,x_0)| \le \alpha(t) M \Lambda^m \qquad \ldots \ldots (50)$$

and

$$|\dot{x}_{m+1}(t,x_0) - \dot{x}_m(t,x_0)| \le 2M\Lambda^m \qquad \ldots \ldots (51)$$

From (50) and (51) we conclude that for any $k \ge 1$, we have the inequalities:

$$|x_{m+k}(t,x_0) - x_m(t,x_0)|_{\circ} \le \frac{MT}{2}\Lambda^m \sum_{j=0}^{k-1} \Lambda^j \quad , \qquad \ldots \ldots (52)$$

and

$$|\dot{x}_{m+k}(t,x_0) - \dot{x}_m(t,x_0)| \le 2M\Lambda^m \sum_{j=0}^{k-1} \Lambda^j \qquad \ldots \ldots (53)$$

for all $t \in R^1$ and $k \ge 1$, where $E$ is identity matrix.

From (52), (53) and the condition (43), the sequence of functions $\{x_m(t,x_0), \dot{x}_m(t,x_0)\}$ is uniformly convergent in the domain $R^1 \times D_f$ as $m \to \infty$. Let

$$\lim_{m\to\infty} x_m(t,x_0) = x^0(t,x_0) \qquad \ldots \ldots (54)$$

and

$$\lim_{m\to\infty} \dot{x}_m(t,x_0) = \dot{x}^0(t,x_0) \qquad \ldots \ldots (55)$$

Since the sequence of functions $x_m(t,x_0)$ and $\dot{x}_m(t,x_0)$ are periodic in t of period T, then the limiting functions $x^0(t,x_0) = x(t,x_0)$ and $\dot{x}^0(t,x_0) = \dot{x}(t,x_0)$ are periodic in t of period T.

Moreover, by the lemma and (52), (53), the inequalities (46), (47) are holds.

Finally, we have to show that $x(t,x_0)$ is unique solution of the system (37). On the contrary, we suppose that there is at least two different solutions $x(t,x_0)$ and $z(t,x_0)$ of (37).

From (45), the following identity holds:

$$\left|x(t,x_0)-z(t,x_0)\right|_{\circ} \le \left(K_1+K_3L_1\frac{\gamma}{\delta}\right)\frac{T}{2}\left|x(t,x_0)-z(t,x_0)\right|_{\circ}+$$

$$+\left(K_2+K_3L_2\frac{\gamma}{\delta}\right)\frac{T}{2}\left|\dot{x}(t,x_0)-\dot{z}(t,x_0)\right|_{\circ} \quad \dots \dots (56)$$

On differentiating (56) we should also obtain

$$\left|\dot{x}(t,x_0)-\dot{z}(t,x_0)\right|_{\circ} \le 2\left(K_1+K_3L_1\frac{\gamma}{\delta}\right)\left\|x(t,x_0)-z(t,x_0)\right|_{\circ}+$$

$$+2\left(K_2+K_3L_2\frac{\gamma}{\delta}\right)\left\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\right|_{\circ} \quad \dots \dots (57)$$

Inequalities (56) and (57) would lead to the estimate

$$\left|x(t,x_0)-z(t,x_0)\right|_{\circ} \le L\Lambda \;, \quad \dots \dots (58)$$

Where

$$L=T\left[\left(K_1+K_3L_1\frac{\gamma}{\delta}\right)\left\|x(t,x_0)-z(t,x_0)\right|_0+\left(K_2+K_3L_2\frac{\gamma}{\delta}\right)\left\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\right|_0\right]$$

and $\Lambda=\left[\left(K_1+K_3L_1\frac{\gamma}{\delta}\right)\frac{T}{2}+2\left(K_2+K_3L_2\frac{\gamma}{\delta}\right)\right]$

By iteration (58) we have

$$\left|x(t,x_0)-z(t,x_0)\right|_{\circ} \le L\Lambda^m \;, \quad \dots \dots (59)$$

But $\Lambda^m \to 0$ as $m\to\infty$, hence proceeding in the last inequality to the limit we obtain that $x(t,x_0)=z(t,x_0)$, which proves the solution is unique, and this completes the proof of theorem 5.

**Existence of Solution of (37)**

The problem of existence solution of the system (37) is uniquely connected with the existence of zeros of the function $\Delta(x_0)$, which has the form:-

$$\Delta(x_0)=\frac{1}{T}\int_0^T f\left(t,x^0(t,x_0),\dot{x}^0(t,x_0),\int_{-\infty}^{t}G(t,s)g\left(s,x^0(s,x_0),\dot{x}^0(s,x_0)\right)ds\right)dt \quad \dots \dots (60)$$

Since this function is approximately determined from the sequence of functions:

$$\Delta_m(x_0)=\frac{1}{T}\int_0^T f\left(t,x_m(t,x_0),\dot{x}_m(t,x_0),\int_{-\infty}^{t}G(t,s)g\left(s,x_m(s,x_0),\dot{x}_m(s,x_0)\right)ds\right)dt \quad \dots \dots (61)$$

Now we prove the following theorem taking into account that the following inequality will be satisfied for all $m\ge 0$.

$$\left|\Delta(x_0) - \Delta_m(x_0)\right|_{\circ} \le \Lambda^{m+1}\left(E - \Lambda\right)^{-1} M \qquad \dots\dots (62)$$

**Theorem 6:**

If the system of equations (37) satisfies the following conditions:

(**i**) The sequence of functions $\Delta_m(x_0)$ has an isolated singular point $x_0 = x^0$ , $\Delta_m(x^0) = 0$ for all $x_0 \in D_f$ and $t \in R^1$.

(**ii**) The index of this point is non-equal's to zero.

(**iii**) There exist a closed convex domain $D_3 \in D_f$ and containing a unique singular point $x^0$, such that on it's boundary $\Gamma_{D_3}$ the following inequality holds:

$$\inf_{x \in \Gamma_{D_3}} \left|\Delta_m(x)\right| \ge \Lambda^{m+1}\left(E - \Lambda\right)^{-1} M ,$$

Where $\Lambda = \left[\left(K_1 + K_3 L_1 \frac{\gamma}{\delta}\right)\frac{T}{2} + 2\left(K_2 + K_3 L_2 \frac{\gamma}{\delta}\right)\right]$ and $m \ge 1$. Then the system (37) has a periodic solution $x = x(t)$ for which $x(0) \in D_3$.

**Proof:**

By using the inequality (62) we can prove the theorem in a similar way to the theorem 5 [ 2 ].

**Remark 2: [ 2 ]**

If the set $D_f$ dose not degenerate to a point, then the $\Delta -$constant of the system (37) may be considered as the function $\Delta = \Delta(0, x_0)$ given on the set $R^1 \times D_f$ . The properties are defined by:

**Theorem 7:**

Let

$$\Delta : D_f \to R^n ,$$

$$\Delta(x_0) = \frac{1}{T}\int_0^T f\left(t, x^0(t,x_0), \dot{x}^0(t,x_0), \int_{-\infty}^{t} G(t,s) g\left(s, x^0(s,x_0), \dot{x}^0(s,x_0)\right) ds\right) dt \qquad \dots\dots (63)$$

where $x^0(t, x_0)$ is the limit of a sequence of periodic functions (60). Then the following inequalities are satisfied:

$$\left|\Delta(x_0)\right|_{\circ} \le M$$

and

$$\left|\Delta(x_0^1) - \Delta(x_0^2)\right|_{\circ} \le E\left[E - \frac{E_1 T}{2} - E_1 E_2 T\left(E - 2E_2\right)^{-1}\right]^{-1}\left[E - 2E_2\left(E - 2E_2\right)^{-1}\right]\left|x_0^1 - x_0^2\right|_{\circ}$$

$$\dots\dots (64)$$

For any $x_0, x_0^1, x_0^2 \in D_f$ and $t \in R^1$, where $E_1 = \left( K_1 + K_3 L_1 \frac{\gamma}{\delta} \right)$ and $E_2 = \left( K_2 + K_3 L_2 \frac{\gamma}{\delta} \right)$.

**Proof:**

From the properties to the function $x^0(t, x_0)$ established by theorem 5, it follows that the function $\Delta(x_0)$ is continuous and bounded in the domain $R^1 \times D_f$.

By using (63), the following inequality holds:

$$\left|\Delta(x_0^1) - \Delta(x_0^2)\right| \le \frac{1}{T}\int_0^T \left| f\left( t, x^0(t, x_0^1), \dot{x}^0(t, x_0^1), \int_{-\infty}^{t} G(t,s) g\left(s, x^0(s, x_0^1), \dot{x}^0(s, x_0^1)\right) ds \right) - \right.$$
$$\left. - f\left( t, x^0(t, x_0^2), \dot{x}^0(t, x_0^2), \int_{-\infty}^{t} G(t,s) g\left(s, x^0(s, x_0^2), \dot{x}^0(s, x_0^2)\right) ds \right) \right| dt$$
$$\le \left( K_1 + K_3 L_1 \frac{\gamma}{\delta} \right) \left| x^0(t, x_0^1) - x^0(t, x_0^2) \right| +$$
$$+ \left( K_2 + K_3 L_2 \frac{\gamma}{\delta} \right) \left| \dot{x}^0(t, x_0^1) - \dot{x}^0(t, x_0^2) \right|$$

and hence

$$\left|\Delta(x_0^1) - \Delta(x_0^2)\right|_\circ \le E_1 \left| x^0(t, x_0^1) - x^0(t, x_0^2) \right|_\circ + E_2 \left| \dot{x}^0(t, x_0^1) - \dot{x}^0(t, x_0^2) \right|_\circ \qquad \dots \dots (65)$$

Where $x^0(t, x_0^1)$ and $x^0(t, x_0^2)$ are the solution of the integral equation:

$$x(t, x_0^k) = x_0^k + \int_0^t \left[ f\left( s, x(s, x_0^k), \dot{x}(s, x_0^k), \int_{-\infty}^{s} G(s,\tau) g\left(\tau, x(\tau, x_0^k), \dot{x}(\tau, x_0^k)\right) d\tau \right) - \right.$$
$$\left. - \frac{1}{T}\int_0^T f\left( s, x(s, x_0^k), \dot{x}(s, x_0^k), \int_{-\infty}^{s} G(s,\tau) g\left(\tau, x(\tau, x_0^k), \dot{x}(\tau, x_0^k)\right) d\tau \right) ds \right] ds \qquad \dots \dots (66)$$

Where k=1,2.

From (66), we find

$$\left| x^0(t, x_0^1) - x^0(t, x_0^2) \right|_\circ \le \left| x_0^1 - x_0^2 \right|_\circ + \frac{E_1 T}{2} \left| x^0(t, x_0^1) - x^0(t, x_0^2) \right|_\circ + \frac{E_2 T}{2} \left| \dot{x}^0(t, x_0^1) - \dot{x}^0(t, x_0^2) \right|_\circ \qquad \dots \dots (67)$$

On differentiating $x^0(t, x_0^1)$ and $x^0(t, x_0^2)$, we get:

$$\left|\dot{x}^0(t,x_0^1)-\dot{x}^0(t,x_0^2)\right|_\circ \le 2E_1\left|x^0(t,x_0^1)-x^0(t,x_0^2)\right|_\circ + 2E_2\left|\dot{x}^0(t,x_0^1)-\dot{x}^0(t,x_0^2)\right|_\circ$$

$$\le 2E_1\left(E-2E_2\right)^{-1}\left|x^0(t,x_0^1)-x^0(t,x_0^2)\right|_\circ \quad \ldots\ldots (68)$$

Using the inequalities (67) and (68) in (65) we have the inequality (64), and this proves the theorem.

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