Hosoya Polynomials of Steiner Distance of an m-Cube and the Square of a Path

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الملخص

تضمن هذا البحث ايجاد متعددات حدود هوسويا نسبة الى مسافة ستينر -3 لبيانات مكعبات Q_m ، و كذلك لمربع درب P_t^2. كما تم الحصول على اقطار ستينر $-n$ لكل من Q_m و P_t^2.

ABSTRACT

The Hosoya polynomials of Steiner **3**-distance of hypercube graphs Q_m, and of the square of a path, P_t^2, are obtained in this paper. The Steiner n-diameters of Q_m and P_t^2 are also obtained.

1. Introduction.

We follow the terminology of [**2,3**]. For a connected graph $G=(V,E)$ of order p, the ***Steiner distance*[4,5]** of a non-empty subset $S \subseteq V(G)$, denoted by $d_G(S)$, or simply $d(S)$, is defined to be the size of the smallest connected subgraph ***T(S)*** of ***G*** that contains ***S***; ***T(S)*** is called a ***Steiner tree*** of ***S***. If $|S|=2$, then ***d(S)*** is the distance between the two vertices of ***S***. For $2 \le n \le p$ and $|S|=n$, the Steiner distance of ***S*** is called ***Steiner n-distance of S*** in ***G***. The ***Steiner n-diameter*** of ***G*** (or the diameter of the Steiner n-distance), denoted by $diam_n^* G$ or $\delta_n^*(G)$, is defined as follows:

$$diam_n^* G = \max\{d_G(S) : S \subseteq V(G), |S| = n\}.$$

Remark 1.1. It is clear that

(1) If $n \ge m$, then $diam_n^* G \ge diam_m^* G$.

(2) If $S' \subseteq S$, then $d_G(S') \le d_G(S)$.

The Steiner n-distance of a vertex $v \in V(G)$, denoted by $W_n^*(v,G)$, is the sum of the Steiner n-distances of all n-subsets containing v. The sum of Steiner n-distances of all n-subsets of $V(G)$ is denoted by $d_n(G)$ or $W_n^*(G)$. It is clear that

$$W_n^*(G) = n^{-1} \sum_{v \in V(G)} W_n^*(v,G). \qquad \text{……. } (\mathbf{1.1})$$

The graph invariant $W_n^*(G)$ ***is called Wiener index of the Steiner n-distance*** of the graph ***G***.

Definition **1.2**[**1**] Let $C_n^*(G,k)$ be the number of n-subsets of distinct vertices of ***G*** with Steiner ***n***-distance ***k***. The graph polynomial defined by

$$H_n^*(G;x) = \sum_{k=n-1}^{\delta_n^*} C_n^*(G,k)x^k , \qquad \text{……. } (\mathbf{1.2})$$

where δ_n^* is the Steiner ***n***-diameter of ***G***; is called the ***Hosoya polynomial of Steiner n-distance of G.***[**1**].

It is clear that

$$W_n^*(G) = \sum_{k=n-1}^{\delta_n^*} kC_n^*(G,k) \qquad \text{……. } (\mathbf{1.3})$$

For $1 \le n \le p$, let $C_n^*(u,G,k)$ be the number of ***n***-subsets ***S*** of distinct vertices of ***G*** containing ***u*** with Steiner ***n***-distance ***k***. It is clear that

$$C_1^*(u,G,0) = 1.$$

Define

$$H_n^*(u,G;x) = \sum_{k=n-1}^{\delta_n^*} C_n^*(u,G,k)x^k . \qquad \text{……. } (\mathbf{1.4})$$

Obviously, for $2 \le n \le p$

$$H_n^*(G;x) = \frac{1}{n} \sum_{u \in V(G)} H_n^*(u,G;x). \qquad \text{……. } (\mathbf{1.5})$$

Ali and Saeed [**1**] were first whom studied this distance-based polynomial for Steiner ***n***-distances, and established Hosoya polynomials of Steiner ***n***-distance for some special graphs and graphs having some kind of regularity, and for Gutman's compound graphs $G_1 \bullet G_2$ and $G_1 : G_2$ in terms of Hosoya polynomials of G_1 and G_2.

In this paper, we obtain the Hosoya polynomial of Steiner **3**-distance of Q_m and P_t^2. Moreover, $diam_n^* Q_m$ and $diam_n^* P_t^2$ are determined.

2. <u>Hypercube Graphs</u> (m-Cube Q_m)

The Cartesian product [**3**] of two connected disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph denoted by $G_1 \times G_2$ with vertex set $V_1 \times V_2$ in which (x_1, y_1) is joined to (x_2, y_2) whenever $\{x_1x_2 \in E_1 and\, y_1 = y_2\}$ or $\{y_1y_2 \in E_2 and\, x_1 = x_2\}$.

If G_1 is a (p_1, q_1)-graph and G_2 is a (p_2, q_2)-graph, then $G_1 \times G_2$ is a $(p_1p_2, p_1q_2+p_2q_1)$-graph.

Now, the graph m-cube Q_m is defined recursively [3] by $Q_1 = K_2$ and $Q_m = Q_{m-1} \times K_2$ for $m \geq 2$. Thus Q_m has 2^m vertices which may be labeled by the binary m-tuples $(s_1, s_2, .., s_m)$ where each s_i is 0 or 1, for $1 \leq i \leq m$. Two vertices of Q_m are adjacent if their binary representations differ at exactly one place.

The diameter of Q_m is m [7], and Q_m is m-regular graph.

We next describe the Steiner n-diameter of the m-cube Q_m.

***Proposition* 2.1.** For $m \geq 2$ and $n \geq 2^m - m + 1$,

$$diam_n^* Q_m = n - 1$$

Proof. Since Q_m is m-connected [3], so the removal of any (m-1)-subset of vertices produces a connected subgraph of order $2^m - m + 1$.

That is for any subset S of order $n \geq 2^m - m + 1$, the induced subgraph $\langle S \rangle$ is connected, which implies that

$$d(S) = n - 1$$

This completes the proof. ■

***Proposition* 2.2.** For $m \geq 2$ and $2 \leq n \leq 2^m - m$,

$$diam_n^* Q_m \geq n$$

Proof. We assume the contrary, that is we let $diam_n^* Q_m < n$, then for any n-subset S of vertices of Q_m, $d(S) = n - 1$. This means that the removal of any $V - S$ subset of vertices produces a connected subgraph of Q_m.

Thus, Q_m is $(|V - S| + 1)$-connected.

But $|V - S| + 1 \geq 2^m - (2^m - m) + 1 = m + 1$

Contradicting the fact that Q_m is m-connected, so, we must have

$$diam_n^* Q_m \geq n.$$ ■

Proposition 2.2 states that for $2 \leq n \leq 2^m - (m - 1)$, n is a lower bound for $diam_n^* Q_m$. We can improve this bound in the next proposition.

***Proposition* 2.3.** For $2 \leq n \leq 2^m - m$

$$diam_n^* Q_m \geq \max\{m, n\}$$

Proof. It is clear that, this is true for m=2 and m=3.

It is known that $diam_2^* Q_m = m$, and $\max\{m, 2\} = m \geq 2$, So it is also true for n=2.

(a) If $\max\{m,n\}=m$, that is $m \geq n$, and if S contains $u_0 = (0,0,...,0)$ and $u_m = (1,1,...,1)$, then $d(u_0, u_m) = m$ and $d(S) \geq m$.

Therefore $diam_n^* Q_m \geq m$.

(b) If $\max\{m,n\}=n$, then by Proposition **2.2**, $diam_n^* Q_m \geq n$.

So, $diam_n^* Q_m \geq \max\{m,n\}$ for $2 \leq n \leq 2^m - m$. ■

In the case of n=3, we have the following result.

Proposition **2.4.** For $m \geq 3$

$$diam_3^* Q_m = m.$$

Proof. The proof is by induction on m.

It is clear that $diam_3^* Q_3 = 3$, thus assume $m \geq 3$. Suppose that the result is true for $m = k(\geq 3)$, and consider $m = k+1$.

Let $S = \{u_1, u_2, u_3\}$ be any 3-subset of vertices of $V(Q_{k+1})$.

We know that

$$Q_{k+1} = Q_k \times K_2.$$

If $S \subseteq V(Q_k)$ or $V(Q'_k)$, then by induction hypothesis $d(S) \leq k$, where Q'_k is the second copy of Q_k. (See Fig. **2.1**).

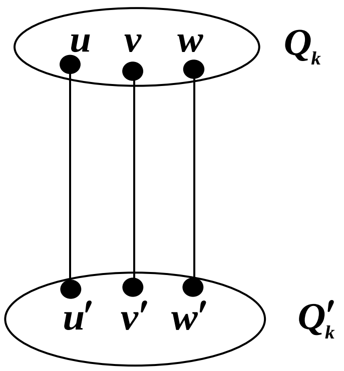

Fig. **2.1.**

Now, let $u_1, u_2 \in V(Q_k)$ and $u_3 \in V(Q'_k)$, and let u'_3 be a vertex in $V(Q_k)$ adjacent to u_3 (see Fig.**2.1**), then

$$d(\{u_1, u_2, u'_3\}) \leq k$$

Thus,

$$diam_3^* Q_{k+1} \leq k+1 = m$$

By Proposition **2.3**, $diam_3^* Q_m \geq m$, because $2 < 2^m - m$ for $m \geq 3$.

Thus,

$$diam_3^* Q_m = m.$$ ■

We next investigate the Hosoya polynomial of Steiner **3**-distance of Q_m, which is obtained as a reduction formula in the following theorem.

Theorem **2.5.** For $m \geq 3$,

$$H_3^*(Q_m;x) = (2+6x)H_3^*(Q_{m-1};x) + 4xH_2^*(Q_{m-1};x),$$

where

$$H_2^*(Q_{m-1};x) = 2^{m-2}(1+x)^{m-1} - 2^{m-2}.$$

Proof. Let S be a **3**-subset of vertices of $V(Q_m)$, and consider $Q_m = Q_{m-1} \times K_2$, assuming that Q_{m-1} and Q'_{m-1} are the two copies of the *(m-1)*-cube in Q_m.

We consider three cases for $d_{Q_m}(S)$.

Case **I.** If $S \subseteq V(Q_{m-1})$ or $V(Q'_{m-1})$, then

$$d_{Q_m}(S) = d_{Q_{m-1}}(S) = d_{Q'_{m-1}}(S).$$

The Hosoya polynomial corresponding to all such S of this case is

$$F_1(x) = 2H_3^*(Q_{m-1};x).$$

Case **II.** Let u,v,w be any **3** vertices of $V(Q_{m-1})$ and u',v',w' are the vertices of $V(Q'_{m-1})$ adjacent respectively to u,v,w as shown in Fig. **2.1** for $k = m-1$.

If $S = \{u,v,w'\}$, $\{u,v',w\}$ $\{u',v,w\}$, $\{u',v',w\}$ $\{u',v,w'\}$or $\{u,v',w'\}$ then

$$d_{Q_m}(S) = 1 + d_{Q_{m-1}}(\{u,v,w\}).$$

Thus, the Hosoya polynomial for all such six possibilities of S is

$$F_2(x) = 6xH_3^*(Q_{m-1};x)$$

Case **III.** If $S = \{u,u',v\}$, $\{u,u',w\}$, $\{u,u',v'\}$ or $\{u,u',w'\}$ then

$$d_{Q_m}(S) = 1 + d_{Q_{m-1}}(S') = 1 + d_{Q_{m-1}}(S''),$$

where $S' = \{u,v\}$ or $\{u,w\}$and $S'' = \{u',v'\}$ or $\{u',w'\}$ and $d_{Q_{m-1}}(S')$ and $d_{Q'_{m-1}}(S'')$ denotes the ordinary distances of $S'$and S''in Q_{m-1} and Q'_{m-1}, respectively.

Thus, the Hosoya polynomial for all such possibilities of S in this case is

$$F_3(x) = 4xH_2^*(Q_{m-1};x).$$

Now, adding the polynomials $F_1(x)$, $F_2(x)$ and $F_3(x)$ we obtain the required reduction formula. ■

Returning to the reduction formula obtained in Theorem **2.5**, we find that $H_3^*(Q_m;x)$ can be simplified as shown in the next corollary.

Corollary **2.6.** For $m \geq 3$

$$H_3^*(Q_m;x) = 4x^2(2+6x)^{m-2} + 4x\sum_{k=1}^{m-2}(2+6x)^{k-1}H_2^*(Q_{m-k};x)$$

Proof. We know that

$$H_3^*(Q_m;x) = (2+6x)H_3^*(Q_{m-1};x) + 4xH_2^*(Q_{m-1};x)$$

$$= (2+6x)[(2+6x)H_3^*(Q_{m-2};x) + 4xH_2^*(Q_{m-2};x)] + 4xH_2^*(Q_{m-1};x)$$

$$= (2+6x)^2 H_3^*(Q_{m-2};x) + 4x[(2+6x)H_2^*(Q_{m-2};x) + H_2^*(Q_{m-1};x)]$$

$$\vdots$$

$$= (2+6x)^{m-2} H_3^*(Q_2;x) + 4x[(2+6x)^{m-3} H_2(Q_{m-(m-2)};x) + (2+6x)^{m-4} H_2(Q_{m-(m-3)};x) + \ldots + H_2(Q_{m-1};x)]$$

It is obvious that $H_3(Q_2;x) = 4x^2$

Hence

$$H_3^*(Q_m;x) = 4x^2(2+6x)^{m-2} + 4x\sum_{r=2}^{m-1}(2+6x)^{m-1-r}H_2(Q_r;x). \quad \blacksquare$$

Next corollary computes the Wiener index of Steiner **3**-distance of Q_m.

Corollary **2.7.** For $m \geq 3$

$$W_3^*(Q_m) = 8^{m-2}(3m+2) + 2^{m-4}\sum_{k=1}^{m-2}4^k\left\{2^{m-k+1}(m-k) + (2^{m-k}-1)(3k+1)\right\}.$$

3. The Square of a Path (P_t^2)

The ***n^th^ power*** G^n [**6**] of a connected graph G has vertex set $V(G)$ and for each distinct vertices u,v of G^n, $uv \in E(G^n)$ whenever

$$1 \leq d_G(u,v) \leq n.$$

It is clear that, if $diamG = n$ then G^n is a complete graph.

In [**7**], W. A. M. Saeed proved that

$$diamG^n = \left\lceil \frac{diamG}{n} \right\rceil.$$

In this section, we consider the square P_t^2 of a path P_t, with respect to Steiner distance. First, we find the Steiner n-diameter.

Proposition **3.1.** For even $t \geq 4$, and for $2 \leq n \leq t$, the Steiner n-diameter of P_t^2 is $\frac{t}{2} - 1 + \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. The graph P_t^2 is shown in Fig.**3.1.**

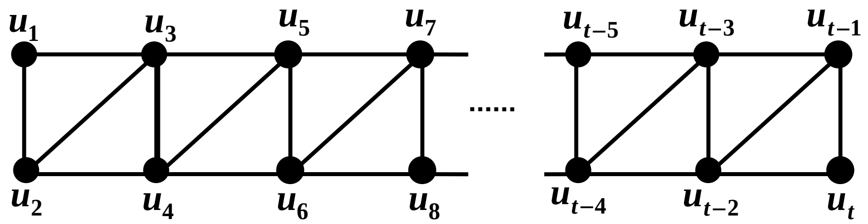

Fig. **3.1.** The square of P_t.

Let $P_t = u_1, u_2, ..., u_t$, then

$$V(P_t^2) = V(P_t) = \{u_1, u_2, ..., u_t\}.$$

If S is an n-subset of vertices of $V(P_t^2)$ such that $d(S)$ is maximum, then S must contain the two vertices u_1 and u_t, the other vertices of S must be the first n-**2** vertices from the sequence (See Fig. **3.1**).

$$u_2, u_3, u_4, ..., u_{t-2}, u_{t-1}.$$

Therefore S contains $\left\lfloor \frac{n-2}{2} \right\rfloor$ vertices from one of the sets $A = \{u_2, u_4, ..., u_{t-2}\}$, $B = \{ u_3, u_5, ..., u_{t-1} \}$ and contains $\left\lceil \frac{n-2}{2} \right\rceil$ vertices from the other set. If S contains $\left\lceil \frac{n-2}{2} \right\rceil$ vertices from A, then $T(S)$ must contain the $u_1 - u_t$ path $u_1, u_2, u_4, ..., u_{t-2}, u_t$, and so S will contain the $\left\lfloor \frac{n-2}{2} \right\rfloor$ vertices from B, and the size of $T(S)$ will be $\frac{t}{2} + \left\lfloor \frac{n-2}{2} \right\rfloor$. But if S contains $\left\lceil \frac{n-2}{2} \right\rceil$ vertices from B, then $T(S)$ must contain the $u_1 - u_t$ path $u_1, u_3, u_5, ..., u_{t-1}, u_t$, and the size of $T(S)$ will also be $\frac{t}{2} + \left\lfloor \frac{n-2}{2} \right\rfloor$.

Hence, the proof of the proposition. ■

Proposition **3.2.** For odd $t \geq 3$, $2 \leq n \leq t$, the Steiner n-diameter of P_t^2 is $\frac{t-3}{2} + \left\lceil \frac{n}{2} \right\rceil$.

Proof. The proof is similar to that of Proposition **3.1**. It is clear that there is exactly one shortest $u_1 - u_t$ path in P_t^2 whose length is $\frac{t-1}{2}$, namely

$u_1, u_3, u_5, ..., u_{t-2}, u_t$. The other $(n-2)$ vertices of the n-subset S are the first $n-2$ from the sequence $u_2, u_3, u_4, ..., u_{t-1}$. Therefore S will contain the first $\left\lceil \frac{n-2}{2} \right\rceil$ vertices from $\{u_2, u_4, ..., u_{t-1}\}$.

Thus S of maximum Steiner n-distance has

$$d(S) = \frac{t-1}{2} + \left\lceil \frac{n-2}{2} \right\rceil = \frac{t-3}{2} + \left\lceil \frac{n}{2} \right\rceil. \qquad \blacksquare$$

Next, we find Hosoya polynomial of the Steiner **3**-distance of the square of a path P_t.

Theorem **3.3.** Let $t = 2s \geq 6$ be an even positive integer, then

$$H_3^*(P_t^2; x) = H_3^*(P_{t-2}^2; x) + F_s(x)$$

where

$$F_s(x) = 2x^2 + 2x^s + \sum_{j=2}^{s-1} [4(x+1)j - 2x - 2]x^j$$

Proof. The graph P_t^2 is shown in Fig.**3.1**; its vertices are relabeled as shown in Fig.**3.2** in order to simplify the derivation of $F_s(x)$.

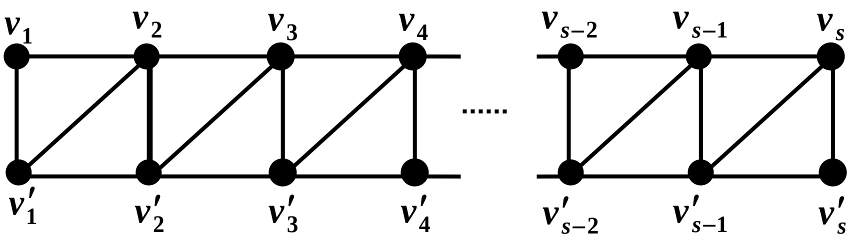

Fig. **3.2.** P_t^2

Let P_{t-2}^2 be obtained from P_t^2 by deleting the two vertices v_s, v'_s

Then

$$H_3^*(P_t^2; x) = H_3^*(P_{t-2}^2; x) + F_s(x),$$

where

$$F_s(x) = \sum_S x^{d(S)} ,$$

in which $|S| = 3$, $S \cap \{v_s, v'_s\} \neq \varphi$ and $S \cap V(P_{t-2}^2) \neq \varphi$.

To find $F_s(x)$ we consider several cases for S.

(1) If $S = \{v_s, v'_s, w\}$, $w \in V(P_{t-2}^2)$, then

$d(S) = s + 1 - i$, when $w = v_i$ *or* v'_i, $1 \leq i \leq s-1$.

Thus, the polynomial corresponding to all such S's of this case is

$$f_1(x)=2\sum_{i=1}^{s-1}x^{s+1-i}=2\sum_{j=2}^{s}x^{j}.$$

(2) If $S=\{v_s,v_i,v_j\}$, $1\le i<j\le s-1$, then

$$d(S)=s-i.$$

It is clear that for each value of i there are $(s-i-1)$ values of j. Thus the corresponding polynomial is $\sum_{i=1}^{s-1}(s-i-1)x^{s-i}$.

The same polynomial is obtained if $S=\{u'_s,u'_i,u'_j\}$.

Therefore, for such **3**-subsets S we get

$$f_2(x)=2\sum_{i=1}^{s-2}(s-i-1)x^{s-i}=2\sum_{j=2}^{s-1}(j-1)x^{j}.$$

(3) If $S=\{v_s,v_i,v'_i\}$ or $\{v'_s,v_i,v'_i\}$, then

$$d(S)=s-i+1,\ 1\le i\le s-1.$$

Thus, the corresponding polynomial is

$$f_3(S)=2\sum_{i=1}^{s-1}x^{s-i+1}=2\sum_{j=1}^{s-1}x^{j+1}.$$

(4) If $S=\{v_s,v_i,v'_j\}$, $1\le i<j\le s-1$, then

$$d(S)=s+1-i.$$

Similarly, if $S=\{v_s,v'_i,v_j\}$, $1\le i<j\le s-1$, then

$$d(S)=s-i$$

Thus, the corresponding polynomial is

$$f_4(x)=\sum_{i=1}^{s-2}(s-i-1)x^{s+1-i}+\sum_{i=1}^{s-2}(s-i-1)x^{s-i}$$
$$=\sum_{j=2}^{s-1}(j-1)(x+1)x^{j}.$$

(5) If $S=\{v'_s,v_i,v'_j\}$, $1\le i<j\le s-1$, then

$$d(S)=s-i+1,$$

and there are $(s-i-1)$ values for j.

Similarly, if $S=\{v'_s,v'_i,v_j\}$ then $d(S)=s-i+1$ for $1\le i<j\le s-1$.

Thus, the polynomial corresponding to all these **3**-subsets is

$$f_5(x)=2\sum_{i=1}^{s-2}(s-i-1)x^{s-i+1}=2\sum_{j=2}^{s-1}(j-1)x^{j+1}.$$

(6) If $S=\{v_s,v'_i,v'_j\}$, $1\le i<j\le s-1$, then

$$d(S)=s-i.$$

The corresponding polynomial is

$$\sum_{i=1}^{s-2}(s-i-1)x^{s-i}.$$

Similarly, if $S=\{v'_s,v_i,v_j\}$, $1\le i<j\le s-1$, then $d(S)=s-i+1$. The corresponding polynomial for such S is

$$\sum_{i=1}^{s-2}(s-i-1)x^{s-i+1}.$$

Thus, the distance polynomial for all these **3**-subsets S in this case is

$$f_6(x)=\sum_{i=1}^{s-2}(s-i-1)x^{s-i}+\sum_{i=1}^{s-2}(s-i-1)x^{s-i+1}$$

$$=\sum_{j=2}^{s-1}(j-1)(x+1)x^{j}$$

These are all possibilities of S. Therefore

$$F_s(x)=\sum_{r=1}^{6}f_r(x)$$

$$=2x^2+2x^s+2\sum_{j=2}^{s-1}x^{j}+2\sum_{j=2}^{s-1}(j-1)x^{j}+2\sum_{j=2}^{s-1}x^{j+1}$$

$$+2\sum_{j=2}^{s-1}(j-1)(x+1)x^{j}+2\sum_{j=2}^{s-1}(j-1)x^{j+1}.$$

Simplifying the above summations we get the reduction formula given in the theorem. ■

The Wiener index of the Steiner **3**-distance of P_t^2 for even t is given in the next corollary.

Corollary **3.4.** For $t=2s\ge 4$,

$$W_3^*(P_t^2)=W_3^*(P_{t-2}^2)+\frac{4}{3}s(s-1)(2s-1).$$

We now consider the square of a path P_t of odd order $t=2s+1$. The next theorem gives us a reduction formula of $H_3^*(P_t^2;x)$.

Theorem **3.5.** For $t=2s\ge 7$, we have

$$H_3^*(P_t^2;x)=H_3^*(P_{t-1}^2;x)+F_s(x),$$

where

$$F_s(x)=x^2+\sum_{j=1}^{s-1}[(x+3)j+x]x^{j+1}.$$

Proof. The graph P_t^2 is shown in Fig. **3.3** where the vertices are labeled as that of Fig. **3.2.**

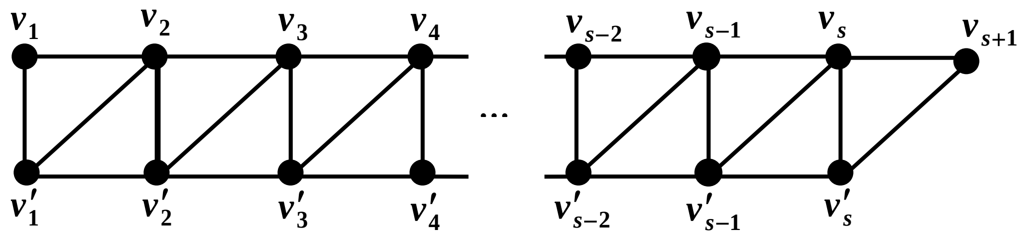

Fig. **3.3.** P_t^2, odd t

$P_{t-1}{}^2$ is obtained from $P_t{}^2$ by removing vertex v_{s+1}. Thus

$$H_3^*(P_t{}^2;x) = H_3^*(P^2{}_{t-1};x) + F_s(x),$$

where

$$F_s(x) = \sum_S x^{d(S)},$$

in which the summation is taken over all **3**-subsets S

$$S = \{v_{s+1}, u_i, u_j\} \text{ for all } u_i, u_j \in V(P_{t-1}^2).$$

We consider the following **5** cases.

(1) If $S = \{v_{s+1}, v_i, v_j\}$, $1 \le i < j \le s$, then

$$d(S) = s + 1 - i.$$

The number of values of j is (s-i) for each values of i. Thus, the polynomial corresponding to such **3**-subsets S of this case is

$$f_1(x) = \sum_{i=1}^{s-1}(s-i)x^{s+1-i} = \sum_{j=1}^{s-1} jx^{j+1}.$$

(2) If $S = \{v_{s+1}, v_i, v'_i\}, 1 \le i \le s$, then

$$d(S) = s + 2 - i.$$

Therefore the corresponding polynomial is

$$f_2(x) = \sum_{i=1}^{s} x^{s+2-i} = x^2 + x^2\sum_{j=1}^{s-1} x^j$$

(3) If $S = \{v_{s+1}, v'_i, v'_j\}$, $1 \le i < j \le s$, then

$$d(S) = s - i + 1,$$

and for each value of i there are (s-i) values for j. Thus, the corresponding polynomial for such case of S is

$$f_3(x) = \sum_{i=1}^{s-1}(s-i)x^{s-i+1} = \sum_{j=1}^{s-1} jx^{j+1}$$

(4) If $S = \{v_{s+1}, v_i, v'_j\}$, $1 \le i < j \le s$, then

$$d(S) = s + 2 - i,$$

and for each value of i there are (s-i) values for j. Thus, the polynomial corresponding to all **3**-subsets S of this case is

$$f_4(x)=\sum_{i=1}^{s-1}(s-i)x^{s+2-i}=x^2\sum_{j=1}^{s-1}jx^j.$$

(5) Finally, If $S=\{v_{s+1},v'_i,v_j\}$, $1\le i<j\le s$, then $d(S)=s+1-i$, and there are (***s-i***) values for ***j*** for each value of ***i***. Therefore, the corresponding polynomial is

$$f_5(x)=\sum_{i=1}^{s-1}(s-i)x^{s+1-i}=\sum_{j=1}^{s-1}jx^{j+1}.$$

Thus,

$$F_s(x)=\sum_{r=1}^{5}f_r(x)=\sum_{j=1}^{s-1}(jx+x^2+jx+x^2j+jx)x^j+x^2$$

$$=x^2+\sum_{j=1}^{s-1}[(x+3)j+x]x^{j+1}.$$ ■

The next corollary gives us the Wiener index of the Steiner **3**-distance of P_t^2 for odd ***t***.

***Corollary* 3.6.** For odd $t=2s+1$, $s\ge 2$, the Wiener index of P_t^2 is

$$W_3^*(P_t^2)=W_3^*(P_{t-1}^2)+\frac{1}{3}(s-1)(4s^2+7s+6)+2.$$

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