Existence and uniqueness of the solution of an integro-differential equation of the second order with boundary conditions

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الخلاصة

يتضمن البحث دراسة وجود وتقارب الحل لنظام من المعادلات تكاملية-التفاضلية اللاخطية من الرتبة الثانية ذات شروط حدودية وذلك باستخدام الطريقة التحليلية-العددية لبناء الحل لنظام المعادلات التفاضلية اللاخطية ذات شروط حدودية لـ Samoilenko A.M. كما تم من خلال هذا البحث تعميم الطريقة في أعلاه و توسيعها.

Abstract

In this paper we investigate the existence and approximation solution for nonlinear system of an integro-differential equation of the second order with boundary conditions by using the numerical-analytic method for investigating a system of nonlinear differential equation with boundary conditions which is given by Samoilenko A. M. And also these investigations lead us to generalized the above method.

1. Introduction:

The numerical- analytic method has been used to study many boundary value problems [1,2,3,4,5,7].

Samoilenko, A. M. and Ronto, N.I. used the numerical- analytic method for investigating a system of nonlinear differential equation with boundary conditions:

$$\frac{dx}{dt} = f(t,x) \qquad \ldots(1.1)$$

$$Ax(0) + Cx(T) = d$$

Where the vector function f (t, x) is continuous in t, x on the domain:

$$(t,x) \in [0,T] \times D \qquad \ldots(1.2)$$

and D is a compact subset of the Euclidean spaces R^n, $A = (A_{ij})$, $C = (C_{ij})$, are positive matrices (n × n), $d \in R^n$.

In this paper we use the numerical-analytic method of a system of nonlinear differential equation which was given by Samoilenko, A. M. Consider the following problem:

$$\left.\begin{array}{ll} \dfrac{d^2x}{dt^2} = f(s, x(s), \dot{x}(s), \int_0^t \phi(s, x(s), \dot{x}(s)ds) & \ldots(1.3) \\ \gamma_1 \dot{x}(0) + \lambda_1 \dot{x}(T) = \delta & \ldots(1.4) \\ \gamma\, x(0) + \lambda\, x(T) = \beta & \ldots(1.5) \end{array}\right\} \ldots(A)$$

The vector functions

$$\left.\begin{array}{l} f(t, x, \dot{x}, y) = (f_1(t, x, \dot{x}, y), \cdots\cdots, f_n(t, x, \dot{x}, y)) \\ \phi(t, x, \dot{x}) = (\phi_1(t, x, \dot{x}), \cdots\cdots\cdots, \phi_n(t, x, \dot{x})) \end{array}\right\} \quad \ldots..(1.6)$$

are continuous vectors on $t, x, \dot{x}, y$ and defined on the domain

$$(t, x, \dot{x}, y) \in [0, T] \times D \times D_1 \times D_2 \quad \ldots.(1.7)$$

where D_1 and D_2 are bounded subset of R^n ,and

$$\gamma = (\gamma_{ij}),\ \lambda = (\lambda_{ij}),\ \gamma_1 = (\gamma_{1ij}),\ \lambda_1 = (\lambda_{1ij}),$$

are positive matrices (n × n), $\beta, \delta \in R^n$.

Let the functions $f(t, x, \dot{x}, y)$ and $\phi(t, x, \dot{x})$ satisfy the following inequalities:

$$\| f(t, x, \dot{x}, y) \| \le M\ , \| \phi(t, x, \dot{x}) \| \le N\ ; \quad \ldots(1.8)$$

$$\| f(t, x_1, \dot{x}_1, y_1) - f(t, x_2, \dot{x}_2, y_2) \| \le K_1 \| x_1 - x_2 \| + K_2 \| \dot{x}_1 - \dot{x}_2 \| + K_3 \| y_1 - y_2 \| \quad \ldots.(1.9)$$

$$\| \phi(t, x_1, \dot{x}_1) - \phi(t, x_2, \dot{x}_2) \| \le L_1 \| x_1 - x_2 \| + L_2 \| \dot{x}_1 - \dot{x}_2 \| \quad \ldots(1.10)$$

for all , $x, x_1, x_2 \in D$, $\dot{x}, \dot{x}_1, \dot{x}_2 \in D_1$, $y, y_1, y_2 \in D_2$, $t \in [0, T]$ and $K_1, K_2, K_3, L_1, L_2, M, N$ are positive constants.

We define the non-empty sets as follows:

$$\begin{array}{l} D_\beta = D - (M \dfrac{T^2}{2} + P_0(x_0)); \\ D_{1\beta} = D_1 - (M \dfrac{T}{2} + P_1(x_0)); \\ D_{2\beta} = D_2 - M_1; \end{array} \quad \ldots.(1.11)$$

and $P_0(x_0) = \left(\dfrac{T + 2T^2}{8} \right) M + \| x_0 \| + \dfrac{T}{4} \| W \| + \| Q \|$, $P_1(x_0) = \dfrac{2}{T} \| x_0 \| + 2TM + \dfrac{2}{T} \| Q \| + \| W \|$

$P_2(x_0) = 2\| x_0 \| + T\| \dot{x}_0 \| + T^2 M + 2\| Q \| + T\| W \|$, $W = \lambda_1^{-1}\delta - \lambda_1^{-1}\gamma_1 \dot{x}_0$, $Q = \lambda^{-1}\beta - \lambda^{-1}\gamma x_0$,

$p(t) = \left(\dfrac{t^2}{T^2} - \dfrac{2t}{T} \right)$, $p_1(t) = \left(\dfrac{t^2}{T} - t \right)$, $p_2(t) = \left(\dfrac{2t}{T^2} - \dfrac{2}{T} \right)$, $p_3(t) = \left(\dfrac{2t}{T^2} - 1 \right)$, $p_4(t) = \left(\dfrac{t^2}{T^2} - \dfrac{t}{T} \right)$

furthermore, we suppose that the largest eigenvalue λ of the following matrix.

$$\Lambda_0 = \begin{pmatrix} \left[M\frac{T^2}{2}+\frac{3T^2}{8}\right](K_1+K_3L_1T) & \left[M\frac{T^2}{2}+\frac{3T^2}{8}\right](K_2+K_3L_2T) \\ \left[\frac{T}{2}+2T\right](K_1+K_3L_1T) & \left[\frac{T}{2}+2T\right](K_2+K_3L_2T) \end{pmatrix}$$

is less than one.i.e.

$$\lambda_1 = 0\,,\ \ \lambda_2 = \left[\left[M\frac{T^2}{2}+\frac{3T^2}{8}\right](K_1+K_3L_1T)+\left[\frac{T}{2}+2T\right](K_2+K_3L_2T)\right] \le 1 \quad \ldots(1.12)$$

LEMMA 1 [6]

Let f (t) be a continuous vector function in the interval $0 \le t \le T$, then

$$\left\| \int_0^t \left(f(s) - \frac{1}{T}\int_0^T f(s)ds \right) ds \right\| \le M\alpha(t)$$

where

$$\alpha(t) = 2t\left(1-\frac{t}{T}\right),\quad M = \max_{t\in[0,T]} \|f(t)\|.$$

LEMMA 2

Let f (t) be a continuous vector function in the interval $0 \le t \le T$, then

$$\left\| \int_0^t \left[\left(\int_0^t f(s)ds \right) - \frac{1}{T}\int_0^T \left(\int_0^t f(s)ds \right) ds \right] ds \right\| \le TM\,\alpha(t) \le M\,\frac{T^2}{2}$$

Where $\left\| \int_0^t f(s)ds \right\| \le \int_0^t \|f(s)\| ds \le tM \le TM$

PROOF:

From the following inequality

$$\left\| \int_0^t \left[\left(\int_0^s f(s)ds \right) - \frac{1}{T}\int_0^T \left(\int_0^s f(s)ds \right) ds \right] ds \right\| \le \left\| \int_0^t \left(\int_0^s f(s)ds \right) ds - \frac{t}{T}\int_0^t \left(\int_0^s f(s)ds \right) ds - \frac{t}{T}\int_t^T \left(\int_0^s f(s)ds \right) ds \right\|$$

$$\le \left\| \left(1-\frac{t}{T}\right)\int_0^t \left(\int_0^s f(s)ds \right) ds - \frac{t}{T}\int_t^T \left(\int_0^s f(s)ds \right) ds \right\|$$

$$\le \left(1-\frac{t}{T}\right)\int_0^t \left\| \int_0^s f(s)ds \right\| ds + \frac{t}{T}\int_t^T \left\| \int_0^s f(s)ds \right\| ds$$

$$\le \left(1-\frac{t}{T}\right)\int_0^t TM\,ds + \frac{t}{T}\int_t^T TM\,ds$$

$$\le TM\alpha(t) \le M\,\frac{T^2}{2}$$

We define the operator L as:

$$L\left(\int_0^t f(s)ds \right) = \int_0^t \left[\left(\int_0^s f(s)ds \right) - \frac{1}{T}\int_0^T \left(\int_0^s f(s)ds \right) ds \right] ds$$

Since f(t) is continuous function on the interval $[0,T]$. then the integral $\left(\int_0^t f(s)ds\right)$ is continuous on the same interval and $L\left(\int_0^t f(s)ds\right)$ is also continuous on the same interval .

By lemma 2 .we get

$$\left\|L\left(\int_0^t f(s)ds\right)\right\| \le \alpha(t)TM \le M\frac{T^2}{2} \quad \text{For all} \quad t\in[0,T]\ ,\ \alpha(t)\le\frac{T}{2}.$$

2. Approximate solution

The investigation of approximate solution of the problem (A) will be introduced by the following theorem.

THEOREM 1

If the system (A) satisfies the inequalities (1.8), (1.9), (1.10) and then the sequence of functions defined by:

$$x_m(t,x_0)=x_0+L(\int_0^t f(s,x_{m-1}(s,x_0),\dot{x}_{m-1}(s,x_0),\int_0^s \phi(s,x_{m-1}(s,x_0),\dot{x}_{m-1}(s,x_0)ds)ds)$$

$$+p_4(t)\int_0^T\int_0^t f(s,x_{m-1}(s,x_0),\dot{x}_{m-1}(s,x_0),\int_0^s \phi(s,x_{m-1}(s,x_0),\dot{x}_{m-1}(s,x_0)ds)dsdt-$$

$$-p_1(t)\int_0^T f(s,x_{m-1}(s,x_0),\dot{x}_{m-1}(s,x_0),\int_0^s \phi(s,x_{m-1}(s,x_0),\dot{x}_{m-1}(s,x_0)ds)ds+p(t)x_0+P_1(t)W+P(t)Q$$

$$x_0(t,x_0)=x_0\ ,\ \frac{dx_m(t,x_0)}{dt}=\dot{x}_m(t,x_0) \qquad \ldots(2.13)$$

As $m\to\infty$

converges uniformly in the domain:

$$(t,x_0)\in[0,T]\times D_\beta \qquad \ldots(2.14)$$

for $x_0\in D_\beta$,

to the function $x_\infty(t,x_0)$ which is satisfying the integral equation

$$x(t,x_0)=x_0+L(\int_0^t f(s,x(s,x_0),\dot{x}(s,x_0),\int_0^s \phi(s,x(s,x_0),\dot{x}(s,x_0)ds)ds)$$

$$+p_4(t)\int_0^T\int_0^t f(s,x(s,x_0),\dot{x}(s,x_0),\int_0^s \phi(s,x(s,x_0),\dot{x}(s,x_0)ds)dsdt- \qquad \ldots(2.15)$$

$$-p_1(t)\int_0^T f(s,x(s,x_0),\dot{x}(s,x_0),\int_0^s \phi(s,x(s,x_0),\dot{x}(s,x_0)ds)ds+p(t)x_0+P_1(t)W+P(t)Q$$

And has a unique solution provided that

$$\|x_\infty(t,x_0)-x_0\|\le M\frac{T^2}{2}+P_0(x_0) \qquad \ldots(2.16)$$

PROOF:

Setting m=1and using lemma 2 in (2.13), we have

$$\|x_1(t,x_0)-x_0\| \le \Big\| x_0 + L(\int_0^t f(s,x_0(s,x_0),\dot{x}_0(s,x_0),\int_0^s \phi(t,x_0(t,x_0),\dot{x}_0(t,x_0)dt)ds)$$

$$+p_4(t)\int_0^T\int_0^t f(s,x_0(s,x_0),\dot{x}_0(s,x_0),\int_0^s \phi(s,x_0(s,x_0),\dot{x}_0(s,x_0)ds)dsdt- \quad ....(2.17)$$

$$-p_1(t)\int_0^T f(s,x_0(s,x_0),\dot{x}_0(s,x_0),\int_0^s \phi(s,x_0(s,x_0),\dot{x}_0(s,x_0)ds)ds + p(t)x_0 + P_1(t)W + P(t)Q - x_0 \Big\|$$

$$\|x_1(t,x_0)-x_0\| \le MT\alpha(t) + P_0(x_0) \le M\frac{T^2}{2} + P_0(x_0) \quad ...(2.18)$$

$x_1(t,x_0)\in D$ for all $,x_0\in D_\beta$

on differentiating $x(t,x_0)$ from (2.15) and taking the norm, we have

$$\|\dot{x}(t,x_0)\| = \Big\| p_2(t)x_0 + \int_0^t \Big(f(s,x(s,x_0),\dot{x}(s,x_0),\int_0^s \phi(s,x(s,x_0),\dot{x}(s,x_0)ds) -$$

$$-\frac{1}{T}\int_0^T f(s,x(s,x_0),\dot{x}(s,x_0),\int_0^s \phi(s,x(s,x_0),\dot{x}(s,x_0)ds)ds \Big) ds +$$

$$+p_2(t)\int_0^T\int_0^t f(s,x(s,x_0),\dot{x}(s,x_0),\int_0^s \phi(s,s,x(s,x_0),\dot{x}(s,x_0)ds)dsdt +$$

$$+(1-\frac{t}{T})\int_0^T f(s,x(s,x_0)\dot{x}(s,x_0),\int_0^s \phi(s,x(s,x_0),\dot{x}(s,x_0)ds)ds$$

$$-p_2(t)Q + p_3(t)W\Big\| \quad ...(2.19)$$

Or

$$\|\dot{x}(t,x_0)\| \le \frac{2}{T}\|x_0\| + M\alpha(t) + 2TM + \frac{2}{T}\|Q\| + \|W\|$$

$$\le M\frac{T}{2} + P_1(x_0) \quad …(2.20)$$

From (2.18),(2.21) we get

$$\|y_1(t,x_0) - y_0(t)\| \le \Big\|\int_0^t \phi(s,x_1(s,x_0),\dot{x}_1(s,x_0))dt - \phi(s,x_0(s,x_0),0)dt\Big\|$$

$$\le \int_0^t \|\phi(s,x_1(s,x_0),\dot{x}_1(s,x_0))dt - \phi(s,x_0(s,x_0),0)\| dt$$

$$\le T\left(L_1\|x_1(t,x_0)-x_0\| + L_2\|\dot{x}_1(t,x_0)\|\right)$$

$$\le T\left(L_1 M\frac{T^2}{2} + P_0(x_0) + L_2 M\frac{T^2}{2} + P_1(x_0)\right)$$

$$\le M_1 \quad …(2.21)$$

where $y_0(t)=\int_0^t \phi(s,x_0(s,x_0),0)ds$, $x_0 \in D_M$ …(2.22)

By lemma 2 and the sequence of the functions (2.14) when m=1 we get $x_1(t,x_0)\in D$ for all $t\in[0,T]$ and for all $x_0 \in D_\beta$

by induction we have:

$$\|x_m(t,x_0)-x_0\| \le M\frac{T^2}{2}+P_0(x_0) \qquad \text{…(2.23)}$$

$$\|y_m(t,x_0)-y_0(t)\| \le M_1 \qquad \text{…(2.24)}$$

$x_m(t,x_0)\in D$ for all $,x_0 \in D_\beta$

and $y_m(t,x_0)\in D_2$ for all $,y_0 \in D_{\beta 2}$

and $y_m(t)=\int_0^t \phi(s,x_m(s,x_0),\dot{x}_m(s,x_0))ds$, m=0,1,2,3…….

Now we prove that the sequence of functions $\{x_m(t,x_0)\}_{m=0}^{\infty}$ converges uniformly in the domain (1.7) ,by (2.13) and lemma 2,we have

$$\|x_{m+1}(t,x_0)-x_m(t,x_0)\| \le T\alpha(t)[(K_1+K_3L_1T)\|x_{m+1}(t,x_0)-x_m(t,x_0)\|$$
$$+(K_2+K_3L_2T)\|\dot{x}_{m+1}(t,x_0)-\dot{x}_m(t,x_0)\|]+\frac{T^2}{8}[(K_1+K_3L_1T)\|x_{m+1}(t,x_0)-x_m(t,x_0)\|$$
$$+(K_2+K_3L_2T)\|\dot{x}_{m+1}(t,x_0)-\dot{x}_m(t,x_0)\|]+\frac{T^2}{4}[(K_1+K_3L_1T)\|x_{m+1}(t,x_0)-x_m(t,x_0)\|$$
$$+(K_2+K_3L_2T)\|\dot{x}_{m+1}(t,x_0)-\dot{x}_m(t,x_0)\|]. \qquad \text{.....(2.25)}$$
$$\le \left[\frac{T^2}{2}+\frac{3T^2}{8}\right][(K_1+K_3L_1T)\|x_{m+1}(t,x_0)-x_m(t,x_0)\|+$$
$$+(K_2+K_3L_2T)\|\dot{x}_{m+1}(t,x_0)-\dot{x}_m(t,x_0)\|]$$

By (1.8) ,(1.9) ,(1.10) and lemma 1, we have

$$\|\dot{x}_{m+1}(t,x_0)-\dot{x}_m(t,x_0)\| \le \alpha(t)[(K_1+K_3L_1T)\|x_{m+1}(t,x_0)-x_m(t,x_0)\|$$
$$+(K_2+K_3L_2T)\|\dot{x}_{m+1}(t,x_0)-\dot{x}_m(t,x_0)\|]+2T[(K_1+K_3L_1T)\|x_{m+1}(t,x_0)-x_m(t,x_0)\|$$
$$\le [\alpha(t)+2T]\ [[(K_1+K_3L_1T)\|x_{m+1}(t,x_0)-x_m(t,x_0)\|+$$
$$+(K_2+K_3L_2T)\|\dot{x}_{m+1}(t,x_0)-\dot{x}_m(t,x_0)\|]]$$
$$\le \left[\frac{T}{2}+2T\right]\ [[(K_1+K_3L_1T)\|x_{m+1}(t,x_0)-x_m(t,x_0)\|+$$
$$+(K_2+K_3L_2T)\|\dot{x}_{m+1}(t,x_0)-\dot{x}_m(t,x_0)\|]] \qquad \text{....(2.26)}$$

Rewrite the inequalities (2.25), (2.26) in vector form as

$$V_{m+1}(t)\le \Lambda(t)V_m(t) \qquad \text{…(2.27)}$$

where

$$V_{m+1}(t)=\begin{pmatrix}\|x_{m+1}(t,x_0)-x_m(t,x_0)\|\\ \|\dot{x}_{m+1}(t,x_0)-\dot{x}_m(t,x_0)\|\end{pmatrix}; \qquad \text{…(2.28)}$$

$$\Lambda(t)=\begin{pmatrix}\left[T\alpha(t)+\frac{3T^2}{8}\right](K_1+K_3L_1T) & \left[T\alpha(t)+\frac{3T^2}{8}\right](K_2+K_3L_2T)\\ \left[\alpha(t)+2T\right](K_1+K_3L_1T) & \left[\alpha(t)+2T\right](K_2+K_3L_2T)\end{pmatrix} \quad \ldots(2.29)$$

it follows from inequality (2.27) that

$$V_{m+1} \le \Lambda_0 V_m \quad \ldots(2.30)$$

where

$$\Lambda_0 = \max_{t\in[0,T]} |\Lambda(t)| \quad \ldots(2.31)$$

From (2.31) and by iteration, we have

$$V_{m+1} \le \Lambda_0^m V_1 \quad \ldots(2.32)$$

which leads to the estimation

$$\sum_{i=1}^{m} V_i \le \sum_{i=1}^{m} \Lambda_0^{i-1} V_1 , \quad \ldots(2.33)$$

since the matrix Λ_0 has eigenvalues (1.12).

and thus the series (2.34) is uniformly convergent, i.e.

$$\lim_{m\to\infty} \sum_{i=1}^{m} \Lambda_0^{i-1} V_1 = \sum_{i=1}^{\infty} \Lambda_0^{i-1} V_1 = (E-\Lambda_0)^{-1} V_1 \quad \ldots(2.34)$$

Where E is the identity matrix.

The limiting relation (2.34) implies to the uniformly convergent of the sequences $[x_m(t,x_0)],[\dot{x}_m(t,x_0)]$ in the domain (2.14)

Let:

$$\left.\begin{aligned}\lim_{m\to\infty} x_m(t,x_0) &= x_\infty(t,x_0)\\ \lim_{m\to\infty} \dot{x}_m(t,x_0) &= \dot{x}_\infty(t,x_0)\end{aligned}\right\} \quad \ldots(2.35)$$

by inequality (2.33), the estimation

$$\begin{pmatrix}\|x_\infty(t,x_0)-x_m(t,x_0)\|\\ \|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\|\end{pmatrix} \le \Lambda_0^m (E-\Lambda_0)^{-1} V_1 \quad \ldots(2.36)$$

is true for m=1,2,3,…

Thus $x_\infty(t,x_0)$ is solutions of the integral equations (2.15).

Finally, we have to show that $x(t,x_0)$ is a unique solution of (A),on the contrary we suppose that there are at least two different solutions $x(t,x_0)$, $z(t,x_0)$ of (A).

From (1.15),(1.16) the following inequalities hold:

$$\|x(t,x_0)-z(t,x_0)\| \le \left[\frac{T^2}{2}+\frac{3T^2}{8}\right]\left[(K_1+K_3L_1T)\|x(t,x_0)-z(t,x_0)\|+\right.$$
$$\left.+(K_2+K_3L_2T)\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|\right]$$

$$\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\| \le \left[\frac{T}{2}+2T\right]\left[(K_1+K_3L_1T)\|x(t,x_0)-z(t,x_0)\|+\right. \qquad \ldots(2.37)$$
$$\left.+(K_2+K_3L_2T)\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|\right]$$

thus:

$$\begin{pmatrix}\|x(t,x_0)-z(t,x_0)\| \\ \|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|\end{pmatrix} \le \Lambda_0 \begin{pmatrix}\|x(t,x_0)-z(t,x_0)\| \\ \|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|\end{pmatrix} \qquad \ldots(2.38)$$

by iteration we find that

$$\begin{pmatrix}\|x(t,x_0)-z(t,x_0)\| \\ \|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|\end{pmatrix} \le \Lambda_0^m \begin{pmatrix}\|x(t,x_0)-z(t,x_0)\| \\ \|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|\end{pmatrix} \qquad \ldots(2.39)$$

Now from the condition (1.12) we have $\Lambda^m \to 0$ as $m \to \infty$,and hence the limit of the last inequality gives $x(t,x_0)=z(t,x_0),\ \dot{x}(t,x_0)=\dot{z}(t,x_0)$ which prove that the solution is unique and this completes the proof of theorem 1..

3. Existence of solution

The problem of existence solution of (A) is uniquely connected with the existence of zeros of the functions, $\Delta(0,x_0)$ which has the form.

$$\Delta(0,x_0)=2x_0+\dot{x}_0-\int_0^T f(s,x_\infty(s,x_0),\dot{x}_\infty(s,x_0),\int_0^s \phi(s,x_\infty(s,x_0),\dot{x}_\infty(s,x_0)ds))ds-$$
$$-\frac{2}{T}\int_0^T\int_0^t f(s,x_\infty(s,x_0),\dot{x}_\infty(s,x_0),\int_0^s \phi(s,x_\infty(s,x_0),\dot{x}_\infty(s,x_0)ds))dsdt+W-\frac{2}{T}Q, \qquad \ldots(3.40)$$

since this functions are approximately determined from the sequence of functions:

$$\Delta_m(0,x_0)=2x_0+\dot{x}_0-\int_0^T f(s,x_m(s,x_0),\dot{x}_m(s,x_0),\int_0^s \phi(s,x_m(s,x_0),\dot{x}_m(s,x_0)ds))ds-$$
$$-\frac{2}{T}\int_0^T\int_0^t f(s,x_m(s,x_0),\dot{x}_m(s,x_0),\int_0^s \phi(s,x_m(s,x_0),\dot{x}_m(s,x_0)ds))dsdt+W-\frac{2}{T}Q \qquad \ldots(3.41)$$

for m=0,1,2,3…

LEMMA 3

Suppose that the conditions of theorem 1 are satisfied, then the following inequality

$$\|\Delta(0,x_0)-\Delta_m(0,x_0)\| \le \left\langle \begin{pmatrix} 2T(K_1+K_3L_1T) \\ 2T(K_2+K_3L_2T)\end{pmatrix}, \Lambda_0^m(E-\Lambda_0)^{-1}V_1 \right\rangle = \sigma_m \qquad (3.42)$$

Satisfied for $m \ge 0$, $y_0 \in D_{\beta 2}, x_0 \in D_\beta$.

PROOF:

From (3.40) and (3.41), we get

$$\|\Delta(0,x_0)-\Delta_m(0,x_0)\| \le 2T(K_1+K_3L_1T)\|x_\infty(t,x_0)-x_m(t,x_0)\|$$
$$+2T(K_2+K_3L_2T)\|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\| \qquad ...(3.43)$$
$$\le \left\langle \begin{pmatrix} 2T(K_1+K_3L_1T) \\ 2T(K_2+K_3L_2T) \end{pmatrix}, \Lambda_0^m(E-\Lambda_0)^{-1}V_1 \right\rangle = \sigma_m$$

Now we prove the following theorem taking into account that the inequality (3.42) will be satisfied for all $m \ge 0$.

THEOREM 2

Suppose that for $m \ge 0$ the sequence of functions $\Delta(0,x_0)$ which are defined in (3.41) satisfies the inequality.

$$\min_{a+h\le x_0\le b-h} \Delta_m(0,x_0) \le -\sigma_m$$
$$\max_{a+h\le x_0\le b-h} \Delta_m(0,x_0) \ge \sigma_m \qquad \ldots(3.44)$$

Where, $h = M\dfrac{T^2}{2} + P_0(x_0)$,

then the boundary value problem (A) has solutions $x = x(t,x_0)$ such that $a+h \le x_0 \le b-h$.

PROOF:

Let x_1, x_2 be any point in the interval $[a+h, b-h]$ such that

$$\Delta(0,x_1) = \min_{a+h\le x_0\le b-h} \Delta(0,x_0)$$
$$a+h \le x_0 \le b-h \qquad \ldots(3.45)$$
$$\Delta(0,x_2) = \max_{a+h\le x_0\le b-h} \Delta(0,x_0)$$

by using the inequalities (3.42),(3.44) we have:

$$\Delta(0,x_1) = \Delta_m(0,x_1) + [\Delta(0,x_1) - \Delta_m(0,x_1)] \le 0$$
$$\Delta(0,x_2) = \Delta_m(0,x_2) + [\Delta(0,x_2) - \Delta_m(0,x_2)] \ge 0 \qquad \ldots(3.46)$$

From the continuity of (3.40) and from (3.46) there exists a singular point $x_\infty = x_0$, $x_\infty \in [x_1, x_2]$, such that: $\Delta(0,x_\infty) = 0$

thus $x = x(t,x_0)$ is a solution of (A).

REMARK

When $R^n = R^1$ i.e. x_0 is a scalar, theorem 2 can be strengthen by omitting the requirement that the singular point should be isolated. (For this remark see [5])

THEOREM 3

If the function $\Delta(0,x_0)$ is defined by

$$\Delta : D_\beta \to R^n$$

$$\Delta(0,x_0)=2x_0+\dot{x}_0-\int_0^T f(s,x_\infty(s,x_0),\dot{x}_\infty(s,x_0),y_\infty(s,x_0))ds-$$

$$-\frac{2}{T}\int_0^T\int_0^t f(s,x_\infty(s,x_0),\dot{x}_\infty(s,x_0),y_\infty(s,x_0))dsdt+(\lambda_1^{-1}\delta-\lambda_1^{-1}\gamma_1)\dot{x}_0-\frac{2}{T}(\lambda^{-1}\beta-\lambda^{-1}\gamma)x_0 \qquad ..(3.47)$$

Where the function $x_\infty(t,x_0)$ is limit of function (2.13) then the inequalities

$$\|\Delta(0,x_0)\|\le MT+\frac{P_2(x_0)}{T} \qquad \ldots(3.48)$$

and

$$\|\Delta(0,x_0^1)-\Delta(0,x_0^2)\|\le\left[E_7+2TE_1E_{11}\left(E_9E_3E_2E_{10}E_7+E_9E_5\right)+\right.$$
$$\left.+2\mathrm{T}E_2E_{12}\left(E_{10}E_4E_1E_9E_5+E_{10}E_9\right)\right]\|x_0^1-x_0^2\|$$
$$+\left[E_5+2TE_1E_{11}\left(E_9E_3E_2E_{10}E_8+E_9E_5\right)+\right.$$
$$\left.+2\mathrm{T}E_2E_{12}\left(E_{10}E_4E_1E_9E_6+E_{10}E_8\right)\right]\|\dot{x}_0^1-\dot{x}_0^2\| \qquad ...(3.49)$$

Are satisfies for $x_0,x_0^1,x_0^2\in D_\beta$

Where, $E_1=(K_1+K_3L_1T)$, $E_2=(K_2+K_3L_2T)$, $E_3=\left(\frac{T^2}{2}+\frac{3T^2}{8}\right)$, $E_4=\left(\frac{T}{2}+2T\right)$

$E_5=\left(1+\|\lambda^{-1}\beta\|\right)$, $E_6=\left(\frac{T}{4}\|\lambda_1^{-1}\delta\|\right)$, $E_7=\left(\frac{2}{T}+\frac{2}{T}\|\lambda^{-1}\beta\|\right)$, $E_8=\left(\|\lambda_1^{-1}\delta\|\right)$,

$E_9=\left(1-E_3E_1\right)^{-1}$, $E_{10}=\left(1-E_4E_2\right)^{-1}$, $E_{11}=\left(1-E_9E_3E_2E_{10}E_4E_1\right)^{-1}$,

$E_{12}=\left(1-E_{10}E_4E_1E_9E_3E_2\right)^{-1}$

PROOF:

From $x_\infty(t,x_0)$ the function $\Delta(0,x_0)$ that continuous and bounded by $M+\frac{B_1}{T}$ $x_0\in D_\beta,\dot{x}_0\in D_{\beta 1}$ and

$$\|\Delta(0,x_0^1)-\Delta(0,x_0^2)\|\le \mathrm{E}_7\|x_0^1-x_0^2\|+E_5\|\dot{x}_0^1-\dot{x}_0^2\|+2TE_1\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\|$$
$$+2\mathrm{TE}_2\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\| ...(3.50)$$

and the functions $x_\infty(t,x_0^1)$, $x_\infty(t,x_0^2)$ are solution of integral equations:

$$x(t,x_0^k)=x_0+L(\int_0^t f(s,x(t,x_0^k),\dot{x}(t,x_0^k),\int_0^s\phi(s,x(s,x_0^k),\dot{x}(s,x_0^k)ds))ds)$$
$$+p_4(t)\int_0^T\int_0^t f(s,x(t,x_0^k),\dot{x}(t,x_0^k),\int_0^s\phi(s,s,x(s,x_0^k),\dot{x}(s,x_0^k)ds)dsdt-$$
$$-p_1(t)\int_0^T f(s,x(s,x_0^k),\dot{x}(s,x_0^k),\int_0^s\phi(s,x(s,x_0^k),\dot{x}(s,x_0^k)ds)ds+$$
$$+p(t)x_0+P_1(t)\lambda_1^{-1}\delta-\lambda_1^{-1}\gamma_1\dot{x}_0+P(t)\lambda^{-1}\beta-\lambda^{-1}\gamma\, x_0 \qquad ...(3.51)$$

By (3.51) and lemma 2 we get

$$\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|\le E_3E_1\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|+$$
$$+E_3E_2\left\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\right\|+E_5\left\|x_0^1-x_0^2\right\|+E_6\left\|\dot{x}_0^1-\dot{x}_0^2\right\| \quad ...(3.52)$$

$$\left\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\right\|\le E_4E_1\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|+$$
$$+E_4E_2\left\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\right\|+E_7\left\|x_0^1-x_0^2\right\|+E_8\left\|\dot{x}_0^1-\dot{x}_0^2\right\| \ldots(3.53)$$

then

$$\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|\le E_9E_3E_2\left\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\right\|+$$
$$+E_9E_5\left\|x_0^1-x_0^2\right\|+E_9E_6\left\|\dot{x}_0^1-\dot{x}_0^2\right\| \quad \ldots(3.54)$$

$$\left\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\right\|\le E_{10}E_4E_1\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|+$$
$$+E_{10}E_7\left\|x_0^1-x_0^2\right\|+E_{10}E_8\left\|\dot{x}_0^1-\dot{x}_0^2\right\| \quad \ldots(3.55)$$

from (3.54) in (3.55) we get

$$\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|\le E_{11}\left(E_9E_3E_2E_{10}E_7+E_9E_5\right)\left\|x_0^1-x_0^2\right\|$$
$$+E_{11}\left(E_9E_3E_2E_{10}E_8+E_9E_5\right)\left\|\dot{x}_0^1-\dot{x}_0^2\right\| \quad ...(3.56)$$

from (3.55) in (3.54) we get

$$\left\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\right\|\le E_{12}\left(E_{10}E_4E_1E_9E_5+E_{10}E_7\right)\left\|x_0^1-x_0^2\right\|$$
$$+E_{12}\left(E_{10}E_4E_1E_9E_6+E_{10}E_8\right)\left\|\dot{x}_0^1-\dot{x}_0^2\right\| \quad ...(3.57)$$

Substituting (3.56) and (3.57) in (3.50). We get (3.49) .

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