ON MAXIMAL CHAINS IN POSETS WITH GROUP ACTIONS

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الملخص

هدفنا الرئيسي من هذا البحث دراسة السلاسل العظمى في بعض المجموعـات المرتبـة جزئيا والتي عليها افعال زمرية ولاحظنا ان هذه الدراسة تعطينا بعض المؤشرات عن نوعية هذه الافعال . لذا سنحتاج الى دراسة سلوك أفعال الزمر على السلاسل.

ABSTRACT

Our main purpose in this work is to study the maximal chains in group-posets to observe that this study gives us indications on the type of some group actions on posets. Therefore we shall study the behavior of the group actions on chains .

§.1 Introduction :

For any group G and any set X , we say that G acts on X from the left if to each $g\in G$ and $x\in X$ there corresponds a unique element in X denoted by ${}^{g}x$ (or some times gx) such that for all $x\in X$ and $g_1,g_2\in G$;

(i) ${}^{e}x = x$ (ii) ${}^{g_1}({}^{g_2}x) = {}^{(g_1g_2)}x$.

Such a set X with a left action of G on it , is called a left G-set , or simply a G-set. [13].

Since the concept of a group action of a group G on a set X began as a group homomorphism ρ : $G \to S_{1x1}$, we can consider any element g in G as a permutation g : $X \to X$ with $g(x) = {}^{g}x$ for all $x\in X$. So this concept can be extended on sets with additional mathematical structure , with ρ : G →isom (X,X) and the isomorphism related to the structure on X.

§.2 Group-posets :

In this section we give the definition of the group actions on posets. This definition is slightly different from the definition given in [5].

Definition (2-1) :

Let G be a group and P a poset , we say that there is a left action of G on P if for every $g\in G$ and $p\in P$ there corresponds a unique element ${}^{g}p\in P$ such that for all $p,q \in P$ and $g,g_1,g_2 \in G$;

(i) ${}^{e}p = p$ (ii) ${}^{g_2}({}^{g_1}p) = {}^{(g_2\, g_1)}p$ (iii) if $p \rangle q$ then ${}^{g}p \rangle {}^{g}q$

Such a poset P with a left action of G on it , is called a left G-poset , (or simply a G-poset) . When condition (iii) is neglected, P is called a G-set . For more details see [7] , [9] and [10] .

Also , for any group G and poset P there is at least the trivial action which defined by : ${}^{g}p = p$ for all g∈G , p∈P.

The following theorem shows that a group action on poset can be defined as a poset automorphism on P.

Theorem (2-2) **:**

Let G acts on the poset P. Then to each g∈G there corresponds an automorphism ρ_g on P defined by :
$\rho_g(p) = {}^{g}p$ for all p∈P. Also , the map $\rho : G \to \mathrm{Aut}(P)$; defined by $\rho(g) = \rho_g$ for all g∈G is a homomorphism called the corresponding homomorphism to the G action on P.

Proof :

Similar to the proof in [9]. ■

Proposition (2-3) **:**

Let E be a G-poset . Then P(E) the family of all subsets of E (the power set of E) is a G-poset with an action defined by ;

$${}^{g}Y = \{x \in E : {}^{g^{-1}}x \in Y\}, \text{for all } g \in G \text{ and } Y \in P(E).$$

Proof :

(i) Let Y∈P(E) , then ; ${}^{e}Y = \{x \in E : {}^{e^{-1}}x \in Y\} = \{x \in E : x \in Y\} = Y$

(ii) For any Y∈P(E) and g_1, g_2∈G;

$$\begin{aligned} {}^{g_2}({}^{g_1}Y) &= \{x \in E : {}^{g_2^{-1}}x \in {}^{g_1}Y\} = \{x \in E : {}^{g_1^{-1}}({}^{g_2^{-1}}x) \in Y\} \\ &= \{x \in E : {}^{g_1^{-1}g_2^{-1}}x \in Y\} = \{x \in E : {}^{(g_2g_1)^{-1}}x \in Y\} = {}^{(g_2g_1)}Y. \end{aligned}$$

(iii) Let X,Y ∈P(E) with Y>X , and let g∈G. So X⊂Y , that is ${}^{g}X \subset {}^{g}Y$.

Hence, ${}^{g}Y > {}^{g}X$. Therefore P(E) is a G-poset. ■

Definition (2-4): [16]

Let P be a G-poset . For each $p \in P$, the set $\{g \in G: {}^{g}p = p\}$ is called the stabilizer of p and denoted by $Stab_G(p)$ or Gp.

Proposition (2-5) : [8]

Let P be a G-poset. Then for any $p \in P$, $Stab_G(p)$ is a subgroup of G.

Proposition (2-6) :

Let P be a G-poset. Then for all $p \in P$.

(1) $G/Stab_G(p)$ is a poset with ;

$$g_1.Stab_G(p) > g_2.Stab_G(p) \text{ if and only if } {}^{g_1}p > {}^{g_2}p$$

(2) $G/Stab_G(p)$ is a G-poset with an action defined by ;

$${}^{t}(g.Stab_G(p)) = (tg).Stab_G(p) \text{ for all } t, g \in G.$$

Proof :

(1)(i) It is obvious that the relation is reflexive .

(ii) Let $g_1.Stab_G(p) \geq g_2.Stab_G(p)$ and $g_2.Stab_G(p) \geq g_1.Stab_G(P)$.

Then ${}^{g_1}p \geq {}^{g_2}p$ and ${}^{g_2}p \geq {}^{g_1}p$. So ${}^{g_1}p = {}^{g_2}p$.

Hence $g_1.Stab_G(p) = g_2.Stab_G(p)$.

(iii) Let $g_1.Stab_G(p) \geq g_2.Stab_G(p)$ and $g_2.Stab_G(p) \geq g_3.Stab_G(P)$.

Then ${}^{g_1}p \geq {}^{g_2}p$ and ${}^{g_2}p \geq {}^{g_3}p$. So ${}^{g_1}p \geq {}^{g_3}p$.

Hence $g_1.Stab_G(p) \geq g_3.Stab_G(p)$.

Therefore $(G/stab_G(p) , \geq)$ is a poset.

(2)(i) ${}^{e}(g.stab_G(p)) = (eg). Stab_G(P) = g.stab_G(p)$,for all $g.stab_G(p) \in G/Stab_G(p)$.

(ii) Let $g.stab_G(p) \in G/Stab_G(p)$ and $t,r \in G$. Then ;

$$^{r}({}^{t}(g.Stab_G(p))) = {}^{r}(tg.Stab_G(p)) = r(tg).Stab_G(p)$$

$$= (rt)g.Stab_G(P) = {}^{rt}(g.Stab_G(p)).$$

(iii) Let $g_1.stab_G(p) > g_2.stab_G(p)$, and $t \in G$.

Then ${}^{g_1}p > {}^{g_2}p$. So ${}^{t}({}^{g_1}p) > {}^{t}({}^{g_2}p$

That is ${}^{tg_1}p > {}^{tg_2}p$. So $tg_1.Stab_G(p) > tg_2.Stab_G(p)$.

Hence, ${}^{t}(g_1Stab_G(p)) > {}^{t}(g_2Stab_G(p))$.

Therefore $G/Stab_G(p)$ is a $G-$poset. ■

Definition (2-7) : [2]

Let P be a poset . We say that the element a of P covers the element b of P if a > b and there is no element $c \in P$ such that $a > c > b$.

Proposition (2-8) :

Let P be a G-poset and $a,b \in P$ with a covers b , then ${}^{g}a$ covers ${}^{g}b$ for all $g \in G$.

Proof :

Suppose that ${}^{g}a$ does not cover ${}^{g}p$, then there exist at least an element $c \in P$ such that ${}^{g}a > c > {}^{g}b$. So ${}^{g^{-1}}({}^{g}a) > {}^{g^{-1}}c > {}^{g^{-1}}({}^{g}b)$.

That is ${}^{g^{-1}}\,{}^{g}a > {}^{g^{-1}}c > {}^{g^{-1}}\,{}^{g}b$. So ${}^{e}a > {}^{g^{-1}}c > {}^{e}b$ .Hence $a > {}^{g^{-1}}c > b$ and this is a contruduction . Therefore ${}^{g}a$ covers ${}^{g}b$. ■

Definition (2-9) : [1]

Let P be a poset . Then the set , $C(P) = \{(a,b) : a \text{ covers } b\} \subset P \times P$, is called the covering poset of P.

Proposition (2-10) :

Let (P,≥) be a poset , then $((P),\underset{C}{\geq})$ is a poset such that : for all (a,b) , $(a',b') \in C(P)$, $(a,b)\underset{C}{\geq}(a',b')$ if and only if $\{(a,b)=(a',b') \text{ or } b \geq a'\}$

Proof :

(i) Let $(a,b)\in C(P)$, then $(a,b)\underset{C}{\geq}(a,b)$.

(ii) Let $(a,b)\underset{C}{\geq}(a',b')$ and $(a',b')\underset{C}{\geq}(a,b)$.

Then either ; $(a,b)=(a',b')$, or $b \geq a'$ and $b' \geq a$.

Now suppose that $b \geq a'$ and $b' \geq a$, then we have $a > b, b \geq a', a' > b'$ and $b' \geq a$.
So, a > a and this is a contradiction . Hence it must be $(a,b)=(a',b')$.

(iii) Let $(a,b)\underset{C}{\geq}(a',b')$ and $(a',b')\underset{C}{\geq}(a'',b'')$.

Then either $(a,b)=(a',b')=(a',b')$, so $(a,b)=(a'',b'')$, or $b \geq a'$ and $b' \geq a''$.

So we have $b \geq a', a' > b'$ and $b' \geq a''$. That is $b \geq a''$. Hence $(a,b)\underset{C}{\geq}(a'',b'')$

Therefore C(P) is a poset. ∎

Theorem (2-11) :

Let P be a G-poset . Then C(P) is also a G-poset with an action defined by; ${}^{g}(a,b)=({}^{g}a,{}^{g}b)$ for all (a,b) ∈ C(P) and g∈G.

Proof :

(i) ${}^{e}(a,b)({}^{e}a,{}^{e}b)=(a,b)$ for all (a,b) ∈C(P).

(ii) ${}^{g_1}({}^{g_2}(a,b))={}^{g_1}({}^{g_2}a,{}^{g_2}b)=({}^{g_1}({}^{g_2}a),{}^{g_1}({}^{g_2}b))$

$$=({}^{g_1g_2}a,{}^{g_1g_2}b)={}^{g_1g_2}(a,b)$$

For all (a,b) ∈ C(P) and $g_1, g_2 \in G$.

(iii) For all $(a,b),(a',b') \in C(P)$ and $g \in G$, with $(a',b') \underset{C}{>} (a,b)$. Then $b' \geq a$

So ${}^{g}b' \geq {}^{g}a$ .Since $(a,b),(a',b') \in C(P)$. Then $({}^{g}a,{}^{g}b),({}^{g}a',{}^{g}b') \in C(P)$

That is $({}^{g}a',{}^{g}b') \underset{C}{>} ({}^{g}a,{}^{g}b)$. Hence ${}^{g}(a',b') \underset{C}{>} {}^{g}(a,b)$.

Therefore C(P) is a G-poset. ■

§3. Group-Chains :

In this section we study the group actions on chains and the behavior of these actions and when the trivial action is the only one.

Definition (3-1) : [2]

A poset P is called a chain (or totally ordered set) if : for all a,b $\in$P : $a \geq b$ or $b \geq a$.

Equivalently, the poset P is called a chain if for every two different elements a,b of P either $a > b$ or $b > a$.

From the definition above , we conclude that every element of a chain covers at most one element and covered at most by one element . Also any chain has at most one maximal element I and one minimal element 0.

Proposition (3-2) : [2]

Any chain X of n elements is isomorphic to the set of natural numbers $\underline{n} = \{1,2,\ldots,n\}$. That is there exists a bijection function $f: X \to \underline{n}$ such that: $f(x_1) \geq f(x_2)$ if and only if $x_1 \geq x_2$.

Theorem (3-3) :

Let $X = \{x_i\}_{i \in I}$ be a G-chain and I be a set of successive integers with $\ldots x_{i-1} < x_i < x_{i+1} < \ldots$

If ${}^{g}x_i = x_j$ then ${}^{g}x_{i+r} = x_{j+r}$ for all $i, j, i+r, j+r \in I$.

Proof :

(i) Let $i+1, j+1 \in I$. Since X is a chain , then x_{i+1} covers x_i and by proposition (2-8), ${}^{g}x_{i+1}$ covers ${}^{g}x_i$.

Since ${}^{g}x_i = x_j$,then x_{j+1} covers ${}^{g}x_i$. So ${}^{g}x_{i+1} = x_{j+1}$.

(ii)Now we shall use the mathematical induction to prove that ${}^{g}x_{i+r} = x_{j+r}$.From (i) we see that ${}^{g}x_{i+1} = x_{j+1}$ for r =1. Suppose ${}^{g}x_{i+n} = x_{j+n}$ for r = n and i+n , j+n $\in$ I. Since X is a chain , then x_{i+n+1} covers x_{i+n}. So ${}^{g}x_{i+n+1}$ covers ${}^{g}x_{i+n}$. Now from ${}^{g}x_{i+n} = {}^{g}x_{j+n}$ we have ${}^{g}x_{i+n+1} = x_{j+n+1}$.

Therefore , ${}^{g}x_{i+r} = x_{j+r}$ for all i , j ,i+r , j+r $\in$ I. ■

Lemma (3-4) :

Let X be a G-chain and g$\in$G . If ${}^{g}x_i = x_t$ and $x_i < x_t$ then ${}^{g^{-1}}x_i < x_i$ for all $x_i \in$X.

Proof :

${}^{g}x_i = x_t \Rightarrow {}^{g^{-1}}({}^{g}x_i) = {}^{g^{-1}}x_t \Rightarrow {}^{g^{-1}g}x_i = {}^{g^{-1}}x_t \Rightarrow {}^{g^{-1}}x_t = x_i$.

Also , $x_i < x_t \Rightarrow {}^{g^{-1}}x_i < {}^{g^{-1}}x_t$. Therefore ${}^{g^{-1}}x_i < x_i$. ■

Proposition (3-5) :

Let X be a G-chain and g$\in$G with $g^{-1} = g$. Then g$\in$Stab$_G(x_i)$ for all $x_i \in$ X.

Proof :

Let ${}^{g}x_i = x_t$ Then $x_i = {}^{g^{-1}} x_t$. So $x_i = {}^{g} x_t$. Suppose that $x_i \neq x_t$. Then either $x_i < x_t$ or $x_t < x_i$ .If $x_i < x_t$ then ${}^{g}x_i < {}^{g} x_t$ .So , $x_t < x_i$. That is a contradiction . Similarly we have a contradiction if $x_t < x_i$.

Hence , since X is a chain , then $x_i = x_t$. So, ${}^{g}x_i = x_i$.

Therefore $g \in Stab_G(x_i)$ for all $x_i \in X$. ■

Theorem (3-6) :

Let (X , ≤) be a G-chain . Then the action of G on X is only the trivial action if X has 0 or I.

Proof :

(i) Let $0 = x_1 \in X$ and $g \in G$. Suppose that ${}^{g}x_1 \neq x_1$, then $x_1 < {}^{g} x_1 [x_1 = 0]$.

Also , ${}^{g^{-1}} x_i < x_1 = 0$. So this is a contradiction . So, ${}^{g}x_1 = x_1$.Now from theorem (3-3) we have ${}^{g}x_i = x_i$ for all $x_i \in X$ and $g \in G$.

(ii) Let $I = x_1 \in X$ and $g \in G$. Suppose that ${}^{g}x_1 \neq x_1$, then ${}^{g}x_1 < x_1 [x_1 = I]$.

Also , $x_1 < {}^{g^{-1}} x_1$. So this is a contradiction .

So , ${}^{g}x_1 = x_1$. Now from theorem (3-3) we have ${}^{g}x_i = x_i$ for all $x_i \in X$ and $g \in G$. ■

The following corollary can be proved directly from the previous theorem , but we will give another proof.

Corollary (3-7):

Let P = $\{p_1,p_2,\ldots,p_n\}$ be a G-chain with $p_1>p_2>\ldots>p_n$. Then P is a trivial G-chain.

Proof :

Suppose that there exists $g \in G$ and $p_i \in P$ such that ${}^g p_i = p_t$ with $t \neq i$. That is ${}^g p_i \neq p_t$. Suppose that $t > i$, then ${}^g p_{i+(n-t)} = p_{t+(n-t)} = p_n$ such that $i+(n-t) \in \{1,2,\ldots,n\}$. Also, ${}^g p_{i+(n-t)+1} = p_{n+1}$ such that $i+(n-t)+1 \in \{1,2,\ldots,n\}$.But $|P| = n$. So $p_{n+1} \notin P$. Hence ${}^g p_{i+(n-t)+1} \neq p_{n+1}$.

Now let ${}^g p_{i+(n-t)+1} = p_r$. Since $p_{i+(n-t)} > p_{i+(n-t)+1}$, then ${}^g p_{i+(n-t)} > {}^g p_{i+(n-t)+1}$. So, $p_n > p_r$ and this is a contradiction . Similarly we have contradiction when $t < i$. Hence $t = i$.

Therefore the G action on P is the trivial action only. ■

§.4 Maximal chains **:**

Finally in this section we will study the maximal chains in group-posets and we shall observe that the study of these kinds of chains give us some indications on the type of some group actions on posets.

Definition (4-1) **:** [3]

Let P be a poset and $X = \{x_i, x_{i+1}, \ldots, x_j\} \subseteq P$ be a chain such that $x_i < x_{i+1} < \ldots < x_j$, then X is called a maximal chain in P if and only if :

(i) There is no element as $c \in P$ such that : $x_i < x_{i+1} < \ldots < c < \ldots < x_j$.

(ii) There is no element as $k \in P$ such that : $k < x_i$ or $x_j < k$.

Proposition (4-2) :

Let P be a G-poset and Y be a maximal chain in P. Then ${}^g Y$ is also a maximal chain in P with $|{}^g Y| = |Y|$.

Proof :

(i) Since Y is a maximal chain in P , so we can say $Y = \{x_i, x_{i+1}, \ldots, x_j\}$ such that x_{r+1} is covers x_r for all $i < r < j$. So , ${}^{g}Y = \{{}^{g}x_i, {}^{g}x_{i+1}, \ldots, {}^{g}x_j\}$ for all $g \in G$. Hence ${}^{g}x_i < {}^{g}x_{i+1} < \ldots < {}^{g}x_j$. Suppose that there exists an element as $c \in P$ such that ${}^{g}x_i < {}^{g}x_{i+1} < \ldots < c < \ldots < {}^{g}x_j$.

Then ${}^{g^{-1}}({}^{g}x_i) < {}^{g^{-1}}({}^{g}x_{i+1}) < \ldots < {}^{g^{-1}}c < \ldots < {}^{g^{-1}}({}^{g}x_j)$.

That is $x_i < x_{i+1} < \ldots < {}^{g^{-1}}c < \ldots < x_j$ and this is a contradiction since Y is a maximal chain.

(ii) suppose that there exists an element $b \in P$ such that $b \leq {}^{g}x_i$ then :

$b \leq {}^{g}x_i \Rightarrow {}^{g^{-1}}b \leq x_i \Rightarrow {}^{g^{-1}}b = x_i \Rightarrow b = {}^{g}x_i$. Similarly,if ${}^{g}x_j \leq a$ then ${}^{g}x_i = a$. Therefore ${}^{g}Y$ is a maximal chain.

Now let the map $f : Y \to {}^{g}Y$ is defined by : $f(y) = {}^{g}y$ for all $y \in Y$.

f is injective map since : $f(y_1) = f(y_2) \Rightarrow {}^{g}y_1 = {}^{g}y_2 \Rightarrow y_1 = y_2$.

Also f is onto since if $x \in {}^{g}Y$ then there exits $y \in Y$ such that $x = {}^{g}y$. Hence , f is bijection and $|Y| = |{}^{g}Y|$. ■

Definition (4-3) **:** [4]

Let P be a poset and $x \in P$. Then the subset C of P is called a cutset of the element x in P if every element of C is not comparable with x and all the maximal chains in P cut with $C \cup \{x\}$. We shall note to this set by cut x.

Theorem (4-4) :

Let P be a G-poset and C is the cutset of $x \in P$. Then ${}^{g}C$ is the cutset of ${}^{g}x$. That is ${}^{g}C = \text{cut}\, {}^{g}x$.

Proof :

Let $y \in cut\, {}^{g}x$ then ${}^{g^{-1}}y$ is not comparable with ${}^{g}x$. So ${}^{g^{-1}}y$ is not comparable with x. That is ${}^{g^{-1}}y \in C$. So ${}^{g}({}^{g^{-1}}y) \in {}^{g}C$. That is $y \in {}^{g}C$.

Hence $cut\, {}^{g}x \subseteq {}^{g}C$.

Now let ${}^{g}s \in {}^{g}C$. Then $s \in C$. So s in not comparable with x. That is ${}^{g}s$ is not comparable with ${}^{g}x$. So ${}^{g}s \in cut\, {}^{g}x$. Therefore ${}^{g}C = cut\, {}^{g}x$ ■

Theorem (4-5) :

Let P be a finite G-poset with P(M)= $\{M_1,M_2,\ldots,M_n\}$ be the set of the maximal chains in P with $|M_i| = |M_j|$ if and only if i = j. Then the trivial action is the only action of G on P.

Proof :

To prove this theorem we must first prove that ${}^{g}M_i = M_i$ for $1 \le i \le n$, after that we must show that ${}^{g}x = x$ for all $x \in M_i$ and $g \in G$

First part :

Our argument proceeds by induction on the number n to prove that ${}^{g}M_i = M_i$ for all $1 \le i \le n$.

Let $|M_1| = r_1$, $|M_2| = r_2, \ldots, |M_n| = r_n$ such that $r_1 < r_2 < \ldots < r_n$.

(i) Let n=2 . That is P(M) = $\{M_1,M_2\}$ with $|M_1| \neq |M_2|$.

Suppose that ${}^{g}M_1 \neq M_1$, then ${}^{g}M_1 = M_2$. So $|{}^{g}M_1| = |M_2| = |M_1|$ and this is a contradiction . Hence ${}^{g}M_1 = M_1$. Similarly we have ${}^{g}M_2 = M_2$.

(ii) Now assume that n=k with ${}^{g}M_i = M_i$ for all $1 \le i \le k$.

Let n=k+1 . Since ${}^{g}M_i = M_i$ for all $1 \le i \le k$.

Suppose that ${}^{g}M_{k+1} \neq M_{k+1}$ then ${}^{g}M_{k+1} = M_j$ for some $1 \leq j \leq k$. So $\left|{}^{g}M_{k+1}\right| = |M_j| = r_j$. But $\left|{}^{g}M_{k+1}\right| = |M_{k+1}| = r_{k+1}$. Hence $r_j = r_{k+1}$, that is j=k+1 , and this is a contradiction since k+1>j . So ${}^{g}M_{k+1} = M_{k+1}$.

Second part:

Since $\{Mi\}_{i=1}^{n}$ is the family of the maximal chains in P , the M_i is a finite maximal chain in P. Using corollary (3-7) we get : ${}^{g}x = x$ for all $x \in M_i$, $g \in G$ with $1 \leq i \leq n$.

Therefore from part one , the action of G on P is the trivial action only. ■

The above theorem is not true when P has two maximal chains M_i,M_j with $|M_i| = |M_j|$ as in the following example .

Example (4-6):

Let P = {a,b,c,d} be a poset with a > b and c > d. So P(M) = $\{M_1, M_2 : M_1 = \{a,b\}\ ,\ M_2 = \{c,d\}\}$. Hence $|M_1| = |M_2|$.

Let G = C_2 ={e,g} with $g^2 = e$, and ${}^{g}a = c, {}^{g}b = d$.

Therefore P is a G-poset and the action is not trivial .

Proposition (4-7) :

Let P(M) = $\{M_1, M_2, \ldots, M_n\}$ be the set of the maximal chains in the G-poset P. Let ${}^{g}M_i = M_t$, then ${}^{g}M_j \neq M_t$ for all $j \neq i$.

Proof :

Suppose that ${}^{g}M_j = M_t$ for some $j \neq i$. Then ${}^{g}M_j = {}^{g}M_i$ for some $j \neq i$. So ${}^{g^{-1}}({}^{g}M_j) = {}^{g^{-1}}({}^{g}M_i)$ for some $j \neq i$.

Hence M_j = M_i for some $j \neq i$. This is a contradiction since $j \neq i$ implies $|P(M)| < n$. Therefore ${}^{g}M_j \neq M_t$ for all $j \neq i$. ■

Proposition (4-8) :

Let P be an injective G-poset , and P(M) = $\{M_1,M_2,\dots,M_n\}$ be the family of the maximal chains in P. Then :

(i) ($|M_i| = |M_j|$ if and only if i = j) ,implies that G = {e}.

(ii) If $|M_1| = |M_2| = \dots = |M_n|$, then $|G| \le n!$.

(iii) If we reordered the maximal chains such that :

$|N_1| = |N_2| = \dots = |N_r| \neq |N_{r+1}| = \dots = |N_t| \neq |N_{t+1}| = \dots = |N_n|$, with $N_i \in P(M)$, $1 \le i \le n$, then : $|G| \le r! x (t-r)! x \dots x (n-k)!$.

Proof :

(i) Since ρ (g) =. ρ_g)(p) = p = I(p) for all p∈P , g∈G , then g∈ ker(ρ).

But ker(ρ) = {e} because ρ is injective .

Then g = e for all g∈G . So G=ker(ρ) = {e}.

(ii) $|M_1| = |M_2| = \dots = |M_n|$. So for all $M_i \in P(M)$ and g∈G there exists some $M_t \in P(M)$ such that ${}^gM_i = M_t$. From proposition (4-7) we have ${}^gM_i \neq M_t$ for all $j \neq i$.

So the Number of permutations on the maximal chains is n!.

Now since P is an injective G-poset, then $|G| \le n!$.

(iii) Applying (ii) on every part of equal parts of :

$|N_1| = |N_2| = \dots = |N_r| \neq |N_{r+1}| = \dots = |N_t| \neq |N_{t+1}| = \dots \neq |N_{k+1}| = \dots = |N_n|$ we get that the number of permutations on the equal parts are , r!, (t-r)!,…,(n-k)! respectively . Using the fundamental principle of counting , the number of the permutations on the maximal chains is r! x(t-r)! x … x(n-k)! .

Since P is an injective G-poset ,then $|G| \le$ r! x(t-r)! x … x (n-k)! . ■

REFERENCES

[1] Behrendt. Gerhard, "Covering Poset" Discrete Math. 71. No.3, (1988), 189-195.

[2] Birkhoff G., "*Lattice Theory*", Amer. Math. Soc. Coll. Pub. Vol. XXV, Third Edition, (1967).

[3] Donnellant T., "*Lattice Theory*", Pergamon Press, (1968).

[4] Hanlon P., "*The Incidence Algebra of a Group Reduced Partially Order Set*" Combinatorial Math. 7, Sprnger. No.829.

[5] Mohammad A.J. & Mohammad S.A. "β-operations on Finite Posets", J. Edu. & Sci., Vol. 19, (1994), 104-114.

[6] Mohammad A.J. & Mohammad S.A. "On Finite, Group-Sets of Finite Groups", J. Edu. & Sci., Vol.22, (1994), 78-84.

[7] Morris I. & Wensley C.D., "Adams Operations and 2-operations in □-Rings", Discrete Mathematics, 50, (1984), 253-270.

[8] Neumann P. M., Stoy G.A. & Thompsone E.C., "*Groups and Geometry*", Vol.I. The Mathematical Institute, Oxford, (1982).

[9] Rose J.S., " *A Course on Group Theory*", Cambridge University Press, Cambridge, (1978).

[10] Solomon L., "The Burnside Algebra of a Finite Group", J. Combine. Theory 2, (1967), 603-615.