The existence and approximation of the periodic solutions for system of second order nonlinear differential equationsby using Lebesgue integrable

R.N.Butris **Merna Adel Aziz**
College of Education College of Computer Science & Math.
Department of mathematics

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الملخص

يتضمن هذا البحث دراسة وجود وتقارب الحـل الـدوري لنظـام مـن المعـادلات التفاضليـة اللاخطيـة مـن الرتبـة الثانيـة وفق مفهـوم ليبيك للتكامــل وبـافتراض أن كـل من الدالتيـن $g(t,x,y,\dot{x},\dot{y}), f(t,x,y,\dot{x},\dot{y})$ قابلتان للقياس عند t ومقيدتين بدالتين قابلتان للتكامـل الليبيكي وذلك باستخدام الطريقة التحليليـة ـ العددية لدراسة الحلول الدورية للمعادلات التفاضلية الاعتيادية اللاخطية لـ ـ Samoilenko.

ABSTRACT

In this paper we study the existence and approximation of the periodicsolutions for system of second order nonlinear differential equations according to Lebesgue integrable concept and by assuming that each of measurable at t and bounded by $f(t,x,y,\dot{x},\dot{y}),g(t,x,y,\dot{x},\dot{y})$ the functions Lebesgue integrable functions numerical -- analytic method has been used to study the periodic The solutions of ordinary differential equations which were introduced by A.M. Samoilenko.

INTRODUCTION

There are many subjects in physics and technology use mathematical methods that depends on the nonlinear differential equations, and it became clear that the existence of the periodic solutions and its algorithm structure from more important problems in the present time, because of the great possibility for employment the electronic computers the numerical analytic method [6] which suggested by Samoilenko to study the periodic solutions for the linear and nonlinear differential equations became the effective mean to find the periodic solutions and its algorithm structure and this method include uniformly sequences of the periodic functions and the result of that study is the using of the periodic solutions on a wide rang in the difference of the new processes in industry and technology as in the studies [1,2,4,5].

We study in this paper the system of the non linear differential equations *with the form*

$$\left.\begin{aligned} \frac{d^2x(t)}{dt^2} &= (A+B(t))(x(t)+\dot{x}(t)) + f(t,x,y,\dot{x},\dot{y}) \\ \frac{d^2y(t)}{dt^2} &= (C+D(t))(y(t)+\dot{y}(t)) + g(t,x,y,\dot{x},\dot{y}) \end{aligned}\right\} \qquad \text{........(1)}$$

represents a closed domain and bounded , the each of

$x \in D \subseteq R^n, D$ *where*

the functions

$$f(t,x,y,\dot{x},\dot{y}) = (f_1(t,x,y,\dot{x},\dot{y}), f_2(t,x,y,\dot{x},\dot{y}),......., f_n(t,x,y,\dot{x},\dot{y}))$$

$$g(t,x,y,\dot{x},\dot{y}) = (g_1(t,x,y,\dot{x},\dot{y}), g_2(t,x,y,\dot{x},\dot{y}),......., g_n(t,x,y,\dot{x},\dot{y}))$$

are defined , vector and continuous in the domain

$$(t,x,y,\dot{x},\dot{y}) \in [0,T] \times D \times D_1 \times D_2 \times D_3 \qquad \text{........(2)}$$

and periodic in t of period T where D_1, D_2, D_3 represents a closed domain and bounded partly from the R^n. Euclidean space and by assumption that the two functions $f(t,x,y,\dot{x},\dot{y})$, $g(t,x,y,\dot{x},\dot{y})$ satisfy the following inequalities:

$$\|f(t,x,y,\dot{x},\dot{y})\| \le m_1(t) \quad , \quad \|g(t,x,y,\dot{x},\dot{y})\| \le m_2(t) \qquad \text{.......(3)}$$

$$\begin{aligned} \|f(t,x_1,y_1,\dot{x}_1,\dot{y}_1) - f(t,x_2,y_2,\dot{x}_2,\dot{y}_2)\| &\le K_1(t)\|x_1-x_2\| + K_2(t)\|y_1-y_2\| \\ &+ K_3(t)\|\dot{x}_1-\dot{x}_2\| + K_4(t)\|\dot{y}_1-\dot{y}_2\| \end{aligned} \qquad \text{.......(4)}$$

$$\begin{aligned} \|g(t,x_1,y_1,\dot{x}_1,\dot{y}_1) - g(t,x_2,y_2,\dot{x}_2,\dot{y}_2)\| &\le L_1(t)\|x_1-x_2\| + L_2(t)\|y_1-y_2\| \\ &+ L_3(t)\|\dot{x}_1-\dot{x}_2\| + L_4(t)\|\dot{y}_1-\dot{y}_2\| \end{aligned} \qquad \text{.......(5)}$$

for all $t \in [0,T]$, $x, x_1, x_2 \in D, y, y_1, y_2 \in D_1, \dot{x}, \dot{x}_1, \dot{x}_2 \in D_2$ and $\dot{y}, \dot{y}_1, \dot{y}_2 \in D_3$
where each of

$$m_1(t), m_2(t), K_1(t), K_2(t), K_3(t), K_4(t), L_1(t), L_2(t), L_3(t), L_4(t)$$

. $0 \le t \le T$ be *Lebesgue integrable functions in the interval and by assuming that,* $A=[A_{ij}], B(t)=[B_{ij}(t)], C=[C_{ij}], D(t)=[D_{ij}(t)]$ *positive matrices are defined in the domain* $-\infty, < 0 \le t \le T < \infty$ *continuous and periodic in t have size (n×n) and satisfy the following inequalities*

$$\|e^{A(t-s)}\| \le Q \quad , \quad \|e^{C(t-s)}\| \le R \qquad \text{..........(6)}$$

$$\|B(t)\| \le H \quad , \quad \|D(t)\| \le J \quad , \quad \|A\| = N_1 \quad , \quad \|C\| = N_2 \qquad \text{..........(7)}$$

$$\|x_\circ\| = \delta_\circ \quad , \quad \|y_\circ\| = \sigma_\circ \qquad \text{...........(8)}$$

where $-\infty, < 0 \le t \le T < \infty$ *, Q , R , H , J,* $\delta_\circ, \sigma_\circ$ *positive constants and let* $0 \le t \le b_1 \le T, 0 \le t \le b_2 \le T$ *where* b_1, b_2 *are two chosen points so that*

$$\int_0^{b_1} m_1(t)\,dt \le c_1 \;,\; \int_0^{b_1} K_1(t)\,dt \le c_2 \le 1 \;,\; \int_0^{b_1} K_2(t)\,dt \le c_3 \le 1$$

$$,\; \int_0^{b_1} K_3(t)\,dt \le c_4 \le 1 \;,\; \int_0^{b_1} K_4(t)\,dt \le c_5 \le 1 \qquad \text{........(9)}$$

$$\int_0^{b_2} m_2(t)\,dt \le \delta_1 \;,\; \int_0^{b_2} L_1(t)\,dt \le \delta_2 \le 1 \;,\; \int_0^{b_2} L_2(t)\,dt \le \delta_3 \le 1$$

$$,\; \int_0^{b_2} L_3(t)\,dt \le \delta_4 \le 1 \;,\; \int_0^{b_2} L_4(t)\,dt \le \delta_5 \le 1 \qquad \text{........(10)}$$

Definition 1 [6] :-

The system of nonlinear differential equations (1) where the right-hand side defined and continuous and periodic in t of period T in the domain (2)
is said to be system – T if
1- The two sets D_f, D_{1f}, D_g, D_{1g} are not empty

$$\left.\begin{aligned} D_f &= D - \frac{T}{2}\left(\frac{T}{2}N + QC_1\right) \ne \phi \\ D_{1f} &= D_2 - \left(\frac{T}{2}N + QC_1\right) \ne \phi \\ D_g &= D_1 - \frac{T}{2}\left(\frac{T}{2}N^* + R\delta_1\right) \ne \phi \\ D_{1g} &= D_3 - \left(\frac{T}{2}N^* + R\delta_1\right) \ne \phi \end{aligned}\right\} \qquad \text{.......(11)}$$

where $N^* = R^2 J\sigma_\circ, N = Q^2 H\delta_\circ$, $\|.\| = \max_{t\in[0.T]} |.|$

2- The greatest eigen value of the matrix

$$\Omega_2 = \begin{bmatrix} \frac{T}{2}(\frac{T}{2}Q(N_1+H)+QC_2) & \frac{T}{2}(\frac{T}{2}QH+QC_4) & \frac{T}{2}QC_3 & \frac{T}{2}QC_5 \\ (\frac{T}{2}Q(N_1+H)+QC_2) & (\frac{T}{2}QH+QC_4) & QC_3 & QC_5 \\ \frac{T}{2}R\delta_2 & \frac{T}{2}R\delta_4 & \frac{T}{2}(\frac{T}{2}R(N_2+J)+R\delta_3) & \frac{T}{2}(\frac{T}{2}RJ+R\delta_5) \\ R\delta_2 & R\delta_4 & (\frac{T}{2}R(N_2+J)+R\delta_3) & (\frac{T}{2}RJ+R\delta_5) \end{bmatrix} \text{is}$$

not greater than 1 that is

$$\lambda_{\max}(\Omega_2) = \frac{\xi_1 + \sqrt{\xi_1^2 + 4\xi_2}}{2} < 1 \qquad ......(12)$$

Where

$$\xi_1 = \frac{T}{2}\rho_1 + \rho_2 + \frac{T}{2}\rho_3 + \rho_4 \quad , \quad \xi_2 = \frac{T^2}{4}v_1 v_3 - \frac{T^2}{4}\rho_1\rho_3 - \frac{T}{2}\rho_1\rho_4 - \frac{T}{2}\rho_2\rho_3 -$$

$$+\frac{T}{2}v_1 v_4 + \frac{T}{2}v_2 v_3 - \rho_2\rho_4 + v_2 v_4$$

$$\rho_1 = \frac{T}{2}Q(N_1+H) + QC_2, \rho_2 = \frac{T}{2}QH + QC_4, \rho_3 = \frac{T}{2}R(N_2+J) + R\delta_3,$$

$$\rho_4 = \frac{T}{2}RJ + R\delta_5 \quad , v_1 = QC_3 \quad , \quad v_2 = QC_5 \quad , \quad v_3 = R\delta_2 \quad , \quad v_4 = R\delta_4$$

Definition 2 [6]:-

The value of intermediary $\mu^* = (\mu_1^*, \mu_2^*)$ in the point $(t, x_\circ, y_\circ)$ which be on it the solution of the system

$$\left.\begin{aligned} \frac{d^2x(t)}{dt^2} &= (A+B(t))(x(t)+\dot{x}(t)) + f(t,x,y,\dot{x},\dot{y}) - \mu_1^* \\ \frac{d^2y(t)}{dt^2} &= (C+D(t))(y(t)+\dot{y}(t)) + g(t,x,y,\dot{x},\dot{y}) - \mu_2^* \end{aligned}\right\} \qquad ....(13)$$

Periodic in t of period T is called constant - Δ^* for the system (1) through the point $t = 0, x = x_\circ, y = y_\circ$ if the intermediary is unique in that point .

Section One : The periodic approximate solution for the system (1)

Lemma 1 :-

Assume that each of $f(t,x,y,\dot{x},\dot{y})$ and $g(t,x,y,\dot{x},\dot{y})$ be vector function and continuous and defined in the interval [0,T] then the inequality

$$\begin{pmatrix} \|M_1^*(t,x_\circ,y_\circ,0,0)\| \\ \|M_2^*(t,x_\circ,y_\circ,0,0)\| \end{pmatrix} \le \begin{pmatrix} \beta_1(t)N+QC_1 \\ \beta_2(t)N^*+R\delta_1 \end{pmatrix} \qquad \text{........(14)}$$

holds for $0 \le t \le T$, $\beta_1(t) \le \frac{T}{2}$, $\beta_2(t) \le \frac{T}{2}$, *where*

$N^* = R^2 J\sigma_\circ , N = Q^2 H\delta_\circ$

$$\beta_1(t) = \left[\frac{t(2e^{\|A\|(T-t)} - e^{\|A\|T} - \|E\|) + T\,(e^{\|A\|T} - e^{\|A\|(T-t)})}{e^{\|A\|T} - \|E\|} \right]$$

$$\beta_2(t) = \left[\frac{t(2e^{\|C\|(T-t)} - e^{\|C\|T} - \|E\|) + T\,(e^{\|C\|T} - e^{\|C\|(T-t)})}{e^{\|C\|T} - \|E\|} \right]$$

$$M_1^*(t,x_\circ,y_\circ,0,0) = \int_0^t e^{A(t-s)} \Big[B(s)(x_\circ e^{At}+0) + f(s,x_\circ,y_\circ,0,0) - \\ - \frac{A}{e^{AT}-E} \int_0^T e^{A(T-s)} [B(s)(x_\circ e^{At}+0) + f(s,x_\circ,y_\circ,0,0)]ds \Big] ds$$

$$M_2^*(t,x_\circ,y_\circ,0,0) = \int_0^t e^{C(t-s)} \Big[D(s)(y_\circ e^{Ct}+0) + g(s,x_\circ,y_\circ,0,0) - \\ - \frac{C}{e^{CT}-E} \int_0^T e^{C(T-s)} [D(s)(y_\circ e^{Ct}+0) + g(s,x_\circ,y_\circ,0,0)]ds \Big] ds$$

Proof

$$\|M_1(t,x_\circ,y_\circ,0,0)\| \le$$

$$\le \left[\|E\| - \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|} \right) \right] \int_0^t \|e^{A(t-s)}\| [\|B(s)\|\|x_\circ\|\|e^{At}\| + \|f(s,x_\circ,y_\circ)\|] ds +$$

$$+ \frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|} \int_t^T \|e^{A(t-s)}\| [\|B(s)\|\|x_\circ\|\|e^{At}\| + \|f(s,x_\circ,y_\circ)\|] ds$$

$$= \beta_1(t)N + QC_1 \qquad \text{........(15)}$$

and also

$\|M_2(t,x_\circ,y_\circ,0,0)\| \le$

$$\le \left[\|E\| - \left(\frac{e^{\|C\|T} - e^{\|C\|(T-t)}}{e^{\|C\|T} - \|E\|}\right)\right]\int_0^t \|e^{C(t-s)}\| [\,\|D(s)\|\,\|y_\circ\|\|e^{Ct}\| + \|g(s,x_\circ,y_\circ)\|]ds +$$

$$+ \frac{e^{\|C\|T} - e^{\|C\|(T-t)}}{e^{\|C\|T} - \|E\|}\int_t^T \|e^{C(t-s)}\| [\|D(s)\|\|y_\circ\|\|e^{Ct}\| + \|g(s,x_\circ,y_\circ)\|]ds$$

$$= \beta_2(t)N^* + R\delta_1 \qquad \text{........(16)}$$

from (15) and (16) we conclude that the inequality (14) holds for

$0 \le t \le T$, $\beta_1(t) \le \frac{T}{2}$, $\beta_2(t) \le \frac{T}{2}$

Now we define the two sequences of the functions

$\{x_m(t,x_\circ,y_\circ)\}_{m=0}^{\infty}$,

$\{y_m(t,x_\circ,y_\circ)\}_{m=0}^{\infty}$ *on the given domain as follows*

$$x_{m+1}(t,x_\circ,y_\circ) = x_\circ e^{At} + L^2\,(e^{A(t-s)}[Ax_m(t,x_\circ,y_\circ) + B(t)(x_m(t,x_\circ,y_\circ) + \dot{x}_m(t,x_\circ,y_\circ)) +$$
$$+ f(t,x_m(t,x_\circ,y_\circ), y_m(t,x_\circ,y_\circ), \dot{x}_m(t,x_\circ,y_\circ), \dot{y}_m(t,x_\circ,y_\circ))] \qquad \text{........(17)}$$

with $x_\circ(t,x_\circ,y_\circ) = x_\circ e^{At}$, $m = 0,1,2$ ....

$$y_{m+1}(t,x_\circ,y_\circ) = y_\circ e^{Ct} + L^2\,(e^{C(t-s)}[Cy_m(t,x_\circ,y_\circ) + D(t)\,(y_m(t,x_\circ,y_\circ) + \dot{y}_m(t,x_\circ,y_\circ)) +$$
$$+ g(t,x_m(t,x_\circ,y_\circ), y_m(t,x_\circ,y_\circ), \dot{x}_m(t,x_\circ,y_\circ), \dot{y}_m(t,x_\circ,y_\circ))] \qquad \text{........(18)}$$

with $y_\circ(t,x_\circ,y_\circ) = y_\circ e^{Ct}$, $m = 0,1,2$.........

also now we define the two sequences of the functions $\{\dot{x}_m(t,x_\circ,y_\circ)\}_{m=0}^{\infty}$,

$\{\dot{y}_m(t,x_\circ,y_\circ)\}_{m=0}^{\infty}$ on the given domain as follows

$$\dot{x}_{m+1}(t,x_\circ,y_\circ) = \int_0^t e^{A(t-s)}[Ax_m(s,x_\circ,y_\circ) + B(s)(x_m(s,x_\circ,y_\circ) + \dot{x}_m(s,x_\circ,y_\circ)) +$$
$$+ f(s,x_m(s,x_\circ,y_\circ), y_m(s,x_\circ,y_\circ), \dot{x}_m(s,x_\circ,y_\circ), \dot{y}_m(s,x_\circ,y_\circ)) -$$
$$- \frac{A}{e^{AT} - E}\int_0^T e^{A(T-s)}[Ax_m(s,x_\circ,y_\circ) + B(s)(x_m(s,x_\circ,y_\circ) + \dot{x}_m(s,x_\circ,y_\circ)) +$$
$$+ f(s,x_m(s,x_\circ,y_\circ), y_m(s,x_\circ,y_\circ), \dot{x}_m(s,x_\circ,y_\circ), \dot{y}_m(s,x_\circ,y_\circ))]ds]ds \qquad \text{........(19)}$$

$$\dot{y}_{m+1}(t,x_\circ,y_\circ) = \int_0^t e^{C(t-s)}[Cy_m(s,x_\circ,y_\circ) + D(s)(y_m(s,x_\circ,y_\circ) + \dot{y}_m(s,x_\circ,y_\circ)) +$$
$$+ g(s,x_m(s,x_\circ,y_\circ), y_m(s,x_\circ,y_\circ), \dot{x}_m(s,x_\circ,y_\circ), \dot{y}_m(s,x_\circ,y_\circ))$$
$$- \frac{C}{e^{CT} - E}\int_0^T e^{C(T-s)}[Cy_m(s,x_\circ,y_\circ) + D(s)(y_m(s,x_\circ,y_\circ) + \dot{y}_m(s,x_\circ,y_\circ)) +$$
$$+ g(s,x_m(s,x_\circ,y_\circ), y_m(s,x_\circ,y_\circ), \dot{x}_m(s,x_\circ,y_\circ), \dot{y}_m(s,x_\circ,y_\circ))]ds]ds \qquad \text{........(20)}$$

We define the two operators L , L^2 as follows :

$$L(e^{A(t-s)}H(t,x_\circ,y_\circ)) = \int_0^t e^{A(t-s)}[H(s,x_\circ,y_\circ)-\Delta_1]ds \quad ,$$

$$L(e^{C(t-s)}Q(t,x_\circ,y_\circ)) = \int_0^t e^{C(t-s)}[Q(s,x_\circ,y_\circ)-\Delta_2]ds \quad ,$$

$$L^2(e^{A(t-s)}H(t,x_\circ,y_\circ)) = \int_0^t [L(e^{A(t-s)}H(s,x_\circ,y_\circ)) - \frac{1}{T}\int_0^T L(e^{A(t-s)}H(s,x_\circ,y_\circ))ds]ds \quad ,$$

$$L^2(e^{C(t-s)}Q(t,x_\circ,y_\circ)) = \int_0^t [L(e^{C(t-s)}Q(s,x_\circ,y_\circ)) - \frac{1}{T}\int_0^T L(e^{C(t-s)}Q(s,x_\circ,y_\circ))ds]ds \quad ,$$

where

$$\Delta_1(x_\circ,y_\circ) = \frac{A}{e^{AT}-E}\int_0^T e^{A(T-t)}[Ax(t,x_\circ,y_\circ)+B(t)(x(t,x_\circ,y_\circ)+\dot{x}(t,x_\circ,y_\circ))+$$
$$+ f(t,x(t,x_\circ,y_\circ),y(t,x_\circ,y_\circ),\dot{x}(t,x_\circ,y_\circ),\dot{y}(t,x_\circ,y_\circ))]dt$$

$$H(t,x_\circ,y_\circ) = Ax(t,x_\circ,y_\circ)+B(t)(x(t,x_\circ,y_\circ)+\dot{x}(t,x_\circ,y_\circ))+$$
$$+ f(t,x(t,x_\circ,y_\circ),y(t,x_\circ,y_\circ),\dot{x}(t,x_\circ,y_\circ),\dot{y}(t,x_\circ,y_\circ))$$

$$\Delta_2(x_\circ,y_\circ) = \frac{C}{e^{CT}-E}\int_0^T e^{C(T-t)}[Cy(t,x_\circ,y_\circ)+D(t)(y(t,x_\circ,y_\circ)+\dot{y}(t,x_\circ,y_\circ))+$$
$$+ g(t,x(t,x_\circ,y_\circ),y(t,x_\circ,y_\circ),\dot{x}(t,x_\circ,y_\circ),\dot{y}(t,x_\circ,y_\circ))]dt$$

$$Q(t,x_\circ,y_\circ) = Cy(t,x_\circ,y_\circ)+D(t)(y(t,x_\circ,y_\circ)+\dot{y}(t,x_\circ,y_\circ))+$$
$$+ g(t,x(t,x_\circ,y_\circ),y(t,x_\circ,y_\circ),\dot{x}(t,x_\circ,y_\circ),\dot{y}(t,x_\circ,y_\circ))$$

It is clear that if each of $e^{A(t-s)}H(t,x_\circ,y_\circ), e^{C(t-s)}Q(t,x_\circ,y_\circ)$ be defined, Continuous and periodic in t on the interval [0,T] then each of $L(e^{A(t-s)}H(t,x_\circ,y_\circ)), L^2(e^{A(t-s)}H(t,x_\circ,y_\circ))$ and $L(e^{C(t-s)}Q(t,x_\circ,y_\circ))$, $L^2(e^{C(t-s)}Q(t,x_\circ,y_\circ))$ are also continuous and periodic in t defined on the same interval By lemma 1 we obtain

$$\|L(e^{A(t-s)}H_\circ(t,x_\circ,y_\circ))\| \le \beta_1(t)\|e^{A(t-s)}\|\|B(t)\|\|x_\circ\|\|e^{At}\| + \|e^{A(t-s)}\|\int_0^t \|f(s,x_\circ,y_\circ,0,0)\|$$
$$\le \frac{T}{2}N + QC_1 \quad , \qquad \text{........(21)}$$

$$\|L(e^{C(t-s)}Q_\circ(t,x_\circ,y_\circ))\| \le \beta_2(t)\|e^{C(t-s)}\|\|D(t)\|\|y_\circ\|\|e^{Ct}\| + \|e^{C(t-s)}\|\int_0^t \|g(s,x_\circ,y_\circ,0,0)\|$$
$$\le \frac{T}{2}N^* + R\delta_1 \quad , \qquad \text{.......(22)}$$

$$\|L^2(e^{A(t-s)}\ H_\circ(t,x_\circ,y_\circ))\| \le \alpha(t)\|L(e^{A(t-s)}\ H_\circ(t,x_\circ,y_\circ))\|$$

$$\le \frac{T}{2}(\frac{T}{2}N + QC_1) \qquad ....(23)$$

$$\|L^2(e^{C(t-s)}Q_\circ(t,x_\circ,y_\circ))\| \le \alpha(t)\|L(e^{C(t-s)}\ Q_\circ(t,x_\circ,y_\circ))\|$$

$$\le \frac{T}{2}(\frac{T}{2}N^* + R\delta_1) \qquad ....(24)$$

for all $t\in[0,T]$, $\beta_1(t)\le\frac{T}{2}$, $\beta_2(t)\le\frac{T}{2}$, $\alpha(t)\le\frac{T}{2}$ where $\alpha(t)=2t(1-\frac{t}{T})$.

Theorem 1:-

Assume that the system (1) satisfy the inequalities (3),(4),(5) and the conditions (11), (12) and if each of $f(t,x,y,\dot{x},\dot{y})$ *and* $g(t,x,y,\dot{x},\dot{y})$ are measurable functions at t in the system (1) and defined in the domain (2) and satisfy the inequalities above and if the inequalities (9),(10) satisfies also and if for the system a periodic solution $x=x(t,x_\circ,y_\circ), y=y(t,x_\circ,y_\circ)$ pass through the point $(t,x_\circ,y_\circ)$ then the two sequences of the functions

$$x_{m+1}(t,x_\circ,y_\circ)=x_\circ e^{At}+L^2(e^{A(t-s)}[Ax_m(t,x_\circ,y_\circ)+B(t)(x_m(t,x_\circ,y_\circ)+\dot{x}_m(t,x_\circ,y_\circ))+$$
$$+f(t,x_m(t,x_\circ,y_\circ),y_m(t,x_\circ,y_\circ),\dot{x}_m(t,x_\circ,y_\circ),\dot{y}_m(t,x_\circ,y_\circ))]) \qquad ....(25)$$

with $x_\circ(t,x_\circ,y_\circ)=x_\circ e^{At}$, $\frac{dx_m}{dt}=\dot{x}_m(t,x_\circ,y_\circ)$, $m=0,1,2,.....$

$$y_{m+1}(t,x_\circ,y_\circ)=y_\circ e^{Ct}+L^2(e^{C(t-s)}[Cy_m(t,x_\circ,y_\circ)+D(t)(y_m(t,x_\circ,y_\circ)+\dot{y}_m(t,x_\circ,y_\circ))+$$
$$+g(t,x_m(t,x_\circ,y_\circ),y_m(t,x_\circ,y_\circ),\dot{x}_m(t,x_\circ,y_\circ),\dot{y}_m(t,x_\circ,y_\circ))]) \qquad ....(26)$$

with $y_\circ(t,x_\circ,y_\circ)=y_\circ e^{Ct}$, $\frac{dy_m}{dt}=\dot{y}_m(t,x_\circ,y_\circ)$, $m=0,1,2,.....$

periodicity in t of period T , its uniformly convergent when $m\to\infty$ in the domain

$$(t,x_\circ,y_\circ)\in[0,T]\times D_f\times D_g \qquad ....(27)$$

from the two functions $x_\infty(t,x_\circ,y_\circ)$ *and* $y_\infty(t,x_\circ,y_\circ)$ are defined in the domain (27) continuous and periodic in t of period T and satisfies the system of integral equations

$$x(t,x_\circ,y_\circ)=x_\circ e^{At}+L^2(e^{A(t-s)}[Ax(t,x_\circ,y_\circ)+B(t)(x(t,x_\circ,y_\circ)+\dot{x}(t,x_\circ,y_\circ))+$$
$$+f(t,x(t,x_\circ,y_\circ),y(t,x_\circ,y_\circ),\dot{x}(t,x_\circ,y_\circ),\dot{y}(t,x_\circ,y_\circ))]) \qquad ....(28)$$
$$y(t,x_\circ,y_\circ)=y_\circ e^{Ct}+L^2(e^{C(t-s)}[Cy(t,x_\circ,y_\circ)+D(t)(y(t,x_\circ,y_\circ)+\dot{y}(t,x_\circ,y_\circ))+$$
$$+g(t,x(t,x_\circ,y_\circ),y(t,x_\circ,y_\circ),\dot{x}(t,x_\circ,y_\circ),\dot{y}(t,x_\circ,y_\circ))]) \qquad ....(29)$$

which are a unique solution for the system (1) and satisfies the following inequality

$$\begin{pmatrix} \|x_\infty(t,x_\circ,y_\circ)-x_m(t,x_\circ,y_\circ)\| \\ \|\dot{x}_\infty(t,x_\circ,y_\circ)-\dot{x}_m(t,x_\circ,y_\circ)\| \\ \|y_\infty(t,x_\circ,y_\circ)-y_m(t,x_\circ,y_\circ)\| \\ \|\dot{y}_\infty(t,x_\circ,y_\circ)-\dot{y}_m(t,x_\circ,y_\circ)\| \end{pmatrix} \le \Omega_2^m(E-\Omega_2)^{-1}V_\circ \qquad ....(30)$$

where $V_\circ=\begin{pmatrix} \frac{T}{2}(\frac{T}{2}N+QC_1) \\ (\frac{T}{2}N+QC_1) \\ \frac{T}{2}(\frac{T}{2}N^*+R\delta_1) \\ (\frac{T}{2}N^*+R\delta_1) \end{pmatrix}$, E the unit matrix

Proof :-

When we look to the two sequences of the functions

$x_1(t,x_\circ,y_\circ),x_2(t,x_\circ,y_\circ),......,x_m(t,x_\circ,y_\circ),.......$

$y_1(t,x_\circ,y_\circ),y_2(t,x_\circ,y_\circ),......,y_m(t,x_\circ,y_\circ),.......$

that are defined in (25),(26) we find that each of the sequences of the functions are defined and continuous in the domain (2) and periodic in t of period T .

By lemma 1 and from (25) when m=0 we obtain

$$\|x_1(t,x_\circ,y_\circ)-x_\circ(t,x_\circ,y_\circ)\|=\|L^2(e^{A(t-s)}H_\circ(t,x_\circ,y_\circ))\|$$

By using (23) we get $\|x_1(t,x_\circ,y_\circ)-x_\circ(t,x_\circ,y_\circ)\|\le\frac{T}{2}(\frac{T}{2}N+QC_1)$

Then $x_1(t,x_\circ,y_\circ)\in D$ for all $x_\circ\in D_f, y_\circ\in D_g$

also from the relation (19) and by lemma 1 when m=0 we obtain

$$\|\dot{x}_1(t,x_\circ,y_\circ)-0\|=\|L(e^{A(t-s)}H_\circ(t,x_\circ,y_\circ))\|$$

By using (21) we get $\|\dot{x}_1(t,x_\circ,y_\circ)-0\|\le\frac{T}{2}N+QC_1$

Then $\dot{x}_1(t,x_\circ,y_\circ)\in D$ for all $x_\circ\in D_f, y_\circ\in D_g$

also from the relation (26) when m=0 we obtain

$$\|y_1(t,x_\circ,y_\circ)-y_\circ(t,x_\circ,y_\circ)\|=\|L^2(e^{C(t-s)}Q_\circ(t,x_\circ,y_\circ))\|$$

By using (24) we get $\|y_1(t,x_\circ,y_\circ)-y_\circ(t,x_\circ,y_\circ)\|\le\frac{T}{2}(\frac{T}{2}N^*+R\delta_1)$

Then $y_1(t,x_\circ,y_\circ)\in D_1$ for all $x_\circ\in D_f, y_\circ\in D_g$

also from the relation (20) when m=0 we obtain

$$\|\dot{y}_1(t,x_\circ,y_\circ)-0\|=\|L(e^{C(t-s)}Q_\circ(t,x_\circ,y_\circ))\|$$

By using (24) we get $\|\dot{y}_1(t,x_\circ,y_\circ)-0\|\le\frac{T}{2}N^*+R\delta_1$

Then $\dot{y}_1(t,x_\circ,y_\circ)\in D_3$ for all $x_\circ\in D_f, y_\circ\in D_g$

and by using the mathematical induction we can prove the truth of the following inequalities for $m\ge1$

$$\left.\begin{aligned}
&\|x_m(t,x_\circ,y_\circ)-x_\circ(t,x_\circ,y_\circ)\|\le\frac{T}{2}(\frac{T}{2}N+QC_1)\\
&\|\dot{x}_m(t,x_\circ,y_\circ)-0\|\le\frac{T}{2}N+QC_1\\
&\|y_m(t,x_\circ,y_\circ)-y_\circ(t,x_\circ,y_\circ)\|\le\frac{T}{2}(\frac{T}{2}N^*+R\delta_1)\\
&\|\dot{y}_m(t,x_\circ,y_\circ)-0\|\le\frac{T}{2}N^*+R\delta_1
\end{aligned}\right\} \quad \text{.......(31)}$$

that is

$x_m(t,x_\circ,y_\circ)\in D, \dot{x}_m(t,x_\circ,y_\circ)\in D_2, y_m(t,x_\circ,y_\circ)\in D_1, \dot{y}_m(t,x_\circ,y_\circ)\in D_3$

for all $x_\circ\in D_f, y_\circ\in D_g$

now we prove that the two sequences

$\{x_m(t,x_\circ,y_\circ)\}_{m=0}^{\infty},\{y_m(t,x_\circ,y_\circ)\}_{m=0}^{\infty}$ are

uniformly convergent in the domain (27) and thereupon each of the two ends for the two sequences are periodicity and continuous in the same domain .

from the previous proof we find

$$\|x_1(t,x_\circ,y_\circ)-x_\circ(t,x_\circ,y_\circ)\|\leq\frac{T}{2}(\frac{T}{2}N+QC_1)\quad,$$

$$\|\dot{x}_1(t,x_\circ,y_\circ)-0\|\leq\frac{T}{2}N+QC_1\quad,$$

$$\|y_1(t,x_\circ,y_\circ)-y_\circ(t,x_\circ,y_\circ)\|\leq\frac{T}{2}(\frac{T}{2}N^*+R\delta_1)\quad,$$

$$\|\dot{y}_1(t,x_\circ,y_\circ)-0\|\leq\frac{T}{2}N^*+R\delta_1\quad.$$

Now when m=1 in (25) and by using (23) we find

$$\|x_2(t,x_\circ,y_\circ)-x_1(t,x_\circ,y_\circ)\|\leq\alpha(t)(\beta_1(t)Q(N_1+H)+QC_2)\|x_1(t,x_\circ,y_\circ)-x_\circ(t,x_\circ,y_\circ)\|+$$
$$+\alpha(t)(\beta_1(t)QH+QC_4)\|\dot{x}_1(t,x_\circ,y_\circ)-0\|+\alpha(t)QC_3\|y_1(t,x_\circ,y_\circ)-y_\circ(t,x_\circ,y_\circ)\|+$$
$$+\alpha(t)QC_5\|\dot{y}_1(t,x_\circ,y_\circ)-0\|$$

also when m=1 in (19) and by using (21) we find

$$\|\dot{x}_2(t,x_\circ,y_\circ)-\dot{x}_1(t,x_\circ,y_\circ)\|\leq(\beta_1(t)Q(N_1+H)+QC_2)\|x_1(t,x_\circ,y_\circ)-x_\circ(t,x_\circ,y_\circ)\|+$$
$$+(\beta_1(t)QH+QC_4)\|\dot{x}_1(t,x_\circ,y_\circ)-0\|+QC_3\|y_1(t,x_\circ,y_\circ)-y_\circ(t,x_\circ,y_\circ)\|+$$
$$+QC_5\|\dot{y}_1(t,x_\circ,y_\circ)-0\|$$

also when m=1 in (26) and by using (24) we find

$$\|y_2(t,x_\circ,y_\circ)-y_1(t,x_\circ,y_\circ)\|\leq\alpha(t)R\delta_2\|x_1(t,x_\circ,y_\circ)-x_\circ(t,x_\circ,y_\circ)\|+$$
$$+\alpha(t)R\delta_4\|\dot{x}_1(t,x_\circ,y_\circ)-0\|+\alpha(t)(\beta_2(t)R(N_2+J)+R\delta_3)\|y_1(t,x_\circ,y_\circ)-y_\circ(t,x_\circ,y_\circ)\|+$$
$$+\alpha(t)(\beta_2(t)RJ+R\delta_5)\|\dot{y}_1(t,x_\circ,y_\circ)-0\|$$

also when m=1 in (22) and by using (20) we find

$$\|\dot{y}_2(t,x_\circ,y_\circ)-\dot{y}_1(t,x_\circ,y_\circ)\|\leq R\delta_2\|x_1(t,x_\circ,y_\circ)-x_\circ(t,x_\circ,y_\circ)\|+$$
$$+R\delta_4\|\dot{x}_1(t,x_\circ,y_\circ)-0\|+(\beta_2(t)R(N_2+J)+R\delta_3)\|y_1(t,x_\circ,y_\circ)-y_\circ(t,x_\circ,y_\circ)\|+$$
$$+(\beta_2(t)RJ+R\delta_5)\|\dot{y}_1(t,x_\circ,y_\circ)-0\|$$

And so by using the mathematical induction we can prove the truth of the following two inequalities :

$$\|x_{m+1}(t,x_\circ,y_\circ)-x_m(t,x_\circ,y_\circ)\| \le$$
$$\le \alpha(t)(\beta_1(t)Q(N_1+H)+QC_2)\|x_m(t,x_\circ,y_\circ)-x_{m-1}(t,x_\circ,y_\circ)\|+$$
$$+\alpha(t)(\beta_1(t)QH+QC_4)\|\dot{x}_m(t,x_\circ,y_\circ)-\dot{x}_{m-1}(t,x_\circ,y_\circ)\|+$$
$$+\alpha(t)QC_3\|y_m(t,x_\circ,y_\circ)-y_{m-1}(t,x_\circ,y_\circ)\|+$$
$$+\alpha(t)QC_5\|\dot{y}_m(t,x_\circ,y_\circ)-\dot{y}_{m-1}(t,x_\circ,y_\circ)\| \quad , \qquad \text{........(32)}$$

$$\|\dot{x}_{m+1}(t,x_\circ,y_\circ)-\dot{x}_m(t,x_\circ,y_\circ)\| \le$$
$$\le (\beta_1(t)Q(N_1+H)+QC_2)\|x_m(t,x_\circ,y_\circ)-x_{m-1}(t,x_\circ,y_\circ)\|+$$
$$+(\beta_1(t)QH+QC_4)\|\dot{x}_m(t,x_\circ,y_\circ)-\dot{x}_{m-1}(t,x_\circ,y_\circ)\|+$$
$$+QC_3\|y_m(t,x_\circ,y_\circ)-y_{m-1}(t,x_\circ,y_\circ)\|+$$
$$+QC_5\|\dot{y}_m(t,x_\circ,y_\circ)-\dot{y}_{m-1}(t,x_\circ,y_\circ)\| \quad , \qquad \text{........(33)}$$

$$\|y_{m+1}(t,x_\circ,y_\circ)-y_m(t,x_\circ,y_\circ)\| \le$$
$$\le \alpha(t)R\delta_2\|x_m(t,x_\circ,y_\circ)-x_{m-1}(t,x_\circ,y_\circ)\|+$$
$$+\alpha(t)R\delta_4\|\dot{x}_m(t,x_\circ,y_\circ)-\dot{x}_{m-1}(t,x_\circ,y_\circ)\|+$$
$$+\alpha(t)(\beta_2(t)R(N_2+J)+R\delta_3)\|y_m(t,x_\circ,y_\circ)-y_{m-1}(t,x_\circ,y_\circ)\|+$$
$$+\alpha(t)(\beta_2(t)RJ+R\delta_5)\|\dot{y}_m(t,x_\circ,y_\circ)-\dot{y}_{m-1}(t,x_\circ,y_\circ)\| \quad , \qquad \text{........(34)}$$

$$\|\dot{y}_{m+1}(t,x_\circ,y_\circ)-\dot{y}_m(t,x_\circ,y_\circ)\| \le$$
$$\le R\delta_2\|x_m(t,x_\circ,y_\circ)-x_{m-1}(t,x_\circ,y_\circ)\|+$$
$$+R\delta_4\|\dot{x}_m(t,x_\circ,y_\circ)-\dot{x}_{m-1}(t,x_\circ,y_\circ)\|+$$
$$+(\beta_2(t)R(N_2+J)+R\delta_3)\|y_m(t,x_\circ,y_\circ)-y_{m-1}(t,x_\circ,y_\circ)\|+$$
$$+(\beta_2(t)RJ+R\delta_5)\|\dot{y}_m(t,x_\circ,y_\circ)-\dot{y}_{m-1}(t,x_\circ,y_\circ)\| \quad , \qquad \text{........(35)}$$

we rewrite (32),(33),(34),(35) with vectors form and on the following mode

$$V_{m+1}(t) \le \Omega_1(t)\, V_m(t) \qquad \text{..........(36)}$$

where

$$V_{m+1}(t)=\begin{pmatrix}\|x_{m+1}(t,x_\circ,y_\circ)-x_m(t,x_\circ,y_\circ)\| \\ \|\dot{x}_{m+1}(t,x_\circ,y_\circ)-\dot{x}_m(t,x_\circ,y_\circ)\| \\ \|y_{m+1}(t,x_\circ,y_\circ)-y_m(t,x_\circ,y_\circ)\| \\ \|\dot{y}_{m+1}(t,x_\circ,y_\circ)-\dot{y}_m(t,x_\circ,y_\circ)\|\end{pmatrix}$$

$$\Omega_1(t)=\begin{bmatrix}\alpha(t)(\beta_1(t)Q(N_1+H)+QC_2) & \alpha(t)(\beta_1(t)QH+QC_4) & \alpha(t)QC_3 & \alpha(t)QC_5 \\ \beta_1(t)Q(N_1+H)+QC_2 & \beta_1(t)QH+QC_4 & QC_3 & QC_5 \\ \alpha(t)R\delta_2 & \alpha(t)R\delta_4 & \alpha(t)(\beta_2(t)R(N_2+J)+R\delta_3) & \alpha(t)(\beta_2(t)RJ+R\delta_5) \\ R\delta_2 & R\delta_4 & \beta_2(t)R(N_2+J)+R\delta_3 & \beta_2(t)RJ+R\delta_5\end{bmatrix}$$

$$V_m(t)=\begin{pmatrix}\|x_m(t,x_\circ,y_\circ)-x_{m-1}(t,x_\circ,y_\circ)\| \\ \|\dot{x}_m(t,x_\circ,y_\circ)-\dot{x}_{m-1}(t,x_\circ,y_\circ)\| \\ \|y_m(t,x_\circ,y_\circ)-y_{m-1}(t,x_\circ,y_\circ)\| \\ \|\dot{y}_m(t,x_\circ,y_\circ)-\dot{y}_{m-1}(t,x_\circ,y_\circ)\|\end{pmatrix}$$

Now we take the maximum value for the two sides of the inequality (36)for

$0\le t\le T$

we find that

$$V_{m+1}\le\Omega_2\,V_m \qquad \text{..........(37)}$$

where $\Omega_2=\max_{t\in[0,T]}\Omega_1(t)$

$$\Omega_2=\begin{bmatrix}\frac{T}{2}(\frac{T}{2}Q(N_1+H)+QC_1) & \frac{T}{2}(\frac{T}{2}QH+QC_4) & \frac{T}{2}QC_3 & \frac{T}{2}QC_5 \\ (\frac{T}{2}Q(N_1+H)+QC_1) & (\frac{T}{2}QH+QC_4) & QC_3 & QC_5 \\ \frac{T}{2}R\delta_2 & \frac{T}{2}R\,\delta_4 & \frac{T}{2}(\frac{T}{2}R(N_2+J)+R\delta_3) & \frac{T}{2}(\frac{T}{2}RJ+R\delta_5) \\ R\delta_2 & R\,\delta_4 & (\frac{T}{2}R(N_2+J)+R\delta_3) & (\frac{T}{2}RJ+R\delta_5)\end{bmatrix}$$

and by repetition (37) we have

$$V_{m+1}\le\Omega_2^m\,V_\circ \qquad \text{..........(38)}$$

also we find

$$\sum_{i=1}^{m}V_i\le\sum_{i=1}^{m}\Omega_2^{i-1}\,V_\circ \qquad \text{..........(39)}$$

since the matrix Ω_2 has the greatest eigen value as in (12) this lead to that the sequence (39) is uniformly convergent that is

$$\lim_{m\to\infty}\sum_{i=1}^{m}\Omega_2^{i-1}V_\circ=\sum_{i=1}^{\infty}\Omega_2^{i-1}V_\circ=(E-\Omega_2)^{-1}V_\circ \qquad \text{..........(40)}$$

and then the relation (40) ascertain on the convergence of sequences of the
functions $[x_m(t,x_\circ,y_\circ),\dot{x}_m(t,x_\circ,y_\circ),y_m(t,x_\circ,y_\circ),\dot{y}_m(t,x_\circ,y_\circ)]$on the domain (27).
We assume that

$$\left.\begin{aligned}\lim_{m\to\infty}x_m(t,x_\circ,y_\circ)&=x_\infty(t,x_\circ,y_\circ)\ ,\\ \lim_{m\to\infty}\dot{x}_m(t,x_\circ,y_\circ)&=\dot{x}_\infty(t,x_\circ,y_\circ)\ ,\\ \lim_{m\to\infty}y_m(t,x_\circ,y_\circ)&=y_\infty(t,x_\circ,y_\circ)\ ,\\ \lim_{m\to\infty}\dot{y}_m(t,x_\circ,y_\circ)&=\dot{y}_\infty(t,x_\circ,y_\circ)\ ,\end{aligned}\right\}\qquad \text{........(41)}$$

since each of the sequences of the functions (25),(19),(26),(20) are continuous and periodic in t of period T then each of the ends for the sequences are continuous and periodic in t of period T and thereupon be

$$x_\infty(t,x_\circ,y_\circ)=x(t,x_\circ,y_\circ)\ ,y_\infty(t,x_\circ,y_\circ)=y(t,x_\circ,y_\circ)$$

$$\dot{x}_\infty(t,x_\circ,y_\circ)=\dot{x}(t,x_\circ,y_\circ)\ ,\dot{y}_\infty(t,x_\circ,y_\circ)=\dot{y}(t,x_\circ,y_\circ)$$

And in addition for that the using of lemma 1 and the relation (40) the inequality (30) satisfies for m ≥ 0
now we prove that $x(t,x_\circ,y_\circ)$, $y(t,x_\circ,y_\circ)$ are unique solution for the system (1) by contradiction .
we assume there at least another two different solutions $x(t,x_\circ,y_\circ),y(t,x_\circ,y_\circ)$,
$r(t,x_\circ,y_\circ),u(t,x_\circ,y_\circ)$ for the system (1) are each of it defined and continuous and periodic in t of period T .
from the form in below :

$$r(t,x_\circ,y_\circ)=x_\circ e^{At}+L^2\,(e^{A(t-s)}[Ar(t,x_\circ,y_\circ)+B(t)(r(t,x_\circ,y_\circ)+\dot{r}(t,x_\circ,y_\circ))+ + f(t,r(t,x_\circ,y_\circ),u(t,x_\circ,y_\circ),\dot{r}(t,x_\circ,y_\circ),\dot{u}(t,x_\circ,y_\circ))])$$

$$H^*(t,x_\circ,y_\circ)=Ar(t,x_\circ,y_\circ)+B(t)(r(t,x_\circ,y_\circ)+\dot{r}(t,x_\circ,y_\circ))+ + f(t,r(t,x_\circ,y_\circ),u(t,x_\circ,y_\circ),\dot{r}(t,x_\circ,y_\circ),\dot{u}(t,x_\circ,y_\circ)) \qquad \text{.........(42)}$$

$$\dot{r}(t,x_\circ,y_\circ)=L(e^{A(t-s)}[Ar(t,x_\circ,y_\circ)+B(t)(r(t,x_\circ,y_\circ)+\dot{r}(t,x_\circ,y_\circ))+ + f(t,r(t,x_\circ,y_\circ),u(t,x_\circ,y_\circ),\dot{r}(t,x_\circ,y_\circ),\dot{u}(t,x_\circ,y_\circ))]) \text{........(43)}$$

$$u(t,x_\circ,y_\circ)=y_\circ e^{Ct}+L^2\,(e^{C(t-s)}[Cu(t,x_\circ,y_\circ)+D(t)\,(u(t,x_\circ,y_\circ)+\dot{u}(t,x_\circ,y_\circ))+$$
$$+\,g(t,r(t,x_\circ,y_\circ),u(t,x_\circ,y_\circ),\dot{r}(t,x_\circ,y_\circ),\dot{u}(t,x_\circ,y_\circ))])$$

$$Q^*(t,x_\circ,y_\circ)=Cu(t,x_\circ,y_\circ)+D(t)(u(t,x_\circ,y_\circ)+\dot{u}(t,x_\circ,y_\circ))+$$
$$+\,g(t,r(t,x_\circ,y_\circ),u(t,x_\circ,y_\circ),\dot{r}(t,x_\circ,y_\circ),\dot{u}(t,x_\circ,y_\circ)) \quad ....(44)$$

$$\dot{u}(t,x_\circ,y_\circ)=L\,(e^{C(t-s)}[Cu(t,x_\circ,y_\circ)+D(t)(u(t,x_\circ,y_\circ)+\dot{u}(t,x_\circ,y_\circ))+$$
$$+\,g(t,r(t,x_\circ,y_\circ),u(t,x_\circ,y_\circ),\dot{r}(t,x_\circ,y_\circ),\dot{u}(t,x_\circ,y_\circ))]) \quad ....(45)$$

We must prove that $r(t,x_\circ,y_\circ)=x(t,x_\circ,y_\circ)$, $\dot{r}(t,x_\circ,y_\circ)=\dot{x}(t,x_\circ,y_\circ)$, $y(t,x_\circ,y_\circ)=u(t,x_\circ,y_\circ), \dot{y}(t,x_\circ,y_\circ)=\dot{u}(t,x_\circ,y_\circ)$ and this from

$$\|x(t,x_\circ,y_\circ)-r(t,x_\circ,y_\circ)\|\le\frac{T}{2}(\frac{T}{2}Q(N_1+H)+QC_2)\|x(t,x_\circ,y_\circ)-r(t,x_\circ,y_\circ)\|+$$
$$+\frac{T}{2}(\frac{T}{2}QH+QC_4)\,\|\dot{x}_1(t,x_\circ,y_\circ)-\dot{r}(t,x_\circ,y_\circ)\|+\frac{T}{2}QC_3\|y(t,x_\circ,y_\circ)-u(t,x_\circ,y_\circ)\|+$$
$$+\frac{T}{2}QC_5\,\|\dot{y}(t,x_\circ,y_\circ)-\dot{u}(t,x_\circ,y_\circ)\| \quad ....(46)$$

$$\|\dot{x}(t,x_\circ,y_\circ)-\dot{r}(t,x_\circ,y_\circ)\|\le(\frac{T}{2}Q(N_1+H)+QC_2)\|x(t,x_\circ,y_\circ)-r(t,x_\circ,y_\circ)\|+$$
$$+(\frac{T}{2}QH+QC_4)\|\dot{x}_1(t,x_\circ,y_\circ)-\dot{r}(t,x_\circ,y_\circ)\|+QC_3\|y(t,x_\circ,y_\circ)-u(t,x_\circ,y_\circ)\|+$$
$$+QC_5\,\|\dot{y}(t,x_\circ,y_\circ)-\dot{u}(t,x_\circ,y_\circ)\| \quad ....(47)$$

$$\|y(t,x_\circ,y_\circ)-u(t,x_\circ,y_\circ)\|\le\frac{T}{2}R\delta_2\,\|x(t,x_\circ,y_\circ)-r(t,x_\circ,y_\circ)\|+$$
$$+\frac{T}{2}R\delta_4\,\|\dot{x}(t,x_\circ,y_\circ)-\dot{r}(t,x_\circ,y_\circ)\|+\frac{T}{2}(\frac{T}{2}R(N_2+J)+R\delta_3)\|y(t,x_\circ,y_\circ)-u(t,x_\circ,y_\circ)\|+$$
$$+\frac{T}{2}(\frac{T}{2}RJ+R\delta_5)\,\|\,\dot{y}(t,x_\circ,y_\circ)-\dot{u}(t,x_\circ,y_\circ)\| \quad ....(48)$$

$$\|\dot{y}(t,x_\circ,y_\circ)-\dot{u}(t,x_\circ,y_\circ)\|\le R\delta_2\,\|x(t,x_\circ,y_\circ)-r(t,x_\circ,y_\circ)\|+$$
$$+R\delta_4\,\|\dot{x}(t,x_\circ,y_\circ)-\dot{r}(t,x_\circ,y_\circ)\|+(\frac{T}{2}R(N_2+J)+R\delta_3)\|y(t,x_\circ,y_\circ)-u(t,x_\circ,y_\circ)\|+$$
$$+(\frac{T}{2}RJ+R\delta_5)\,\|\,\dot{y}(t,x_\circ,y_\circ)-\dot{u}(t,x_\circ,y_\circ)\| \quad ....(49)$$

From (46),(47),(48),(49) we find

$$\begin{pmatrix} \|x(t,x_\circ,y_\circ)-r(t,x_\circ,y_\circ)\| \\ \|\dot{x}(t,x_\circ,y_\circ)-\dot{r}(t,x_\circ,y_\circ)\| \\ \|y(t,x_\circ,y_\circ)-u(t,x_\circ,y_\circ)\| \\ \|\dot{y}(t,x_\circ,y_\circ)-\dot{u}(t,x_\circ,y_\circ)\| \end{pmatrix} \le \Omega_2 \begin{pmatrix} \|x(t,x_\circ,y_\circ)-r(t,x_\circ,y_\circ)\| \\ \|\dot{x}(t,x_\circ,y_\circ)-\dot{r}(t,x_\circ,y_\circ)\| \\ \|y(t,x_\circ,y_\circ)-u(t,x_\circ,y_\circ)\| \\ \|\dot{y}(t,x_\circ,y_\circ)-\dot{u}(t,x_\circ,y_\circ)\| \end{pmatrix} \qquad \text{........(50)}$$

and by repetition (50) we obtain

$$\begin{pmatrix} \|x(t,x_\circ,y_\circ)-r(t,x_\circ,y_\circ)\| \\ \|\dot{x}(t,x_\circ,y_\circ)-\dot{r}(t,x_\circ,y_\circ)\| \\ \|y(t,x_\circ,y_\circ)-u(t,x_\circ,y_\circ)\| \\ \|\dot{y}(t,x_\circ,y_\circ)-\dot{u}(t,x_\circ,y_\circ)\| \end{pmatrix} \le \Omega_2^m \begin{pmatrix} \|x(t,x_\circ,y_\circ)-r(t,x_\circ,y_\circ)\| \\ \|\dot{x}(t,x_\circ,y_\circ)-\dot{r}(t,x_\circ,y_\circ)\| \\ \|y(t,x_\circ,y_\circ)-u(t,x_\circ,y_\circ)\| \\ \|\dot{y}(t,x_\circ,y_\circ)-\dot{u}(t,x_\circ,y_\circ)\| \end{pmatrix}$$

but from the condition (12) we get $\Omega_2^m \to 0$ when $m \to \infty$ and then we have from the last inequality that

$x(t,x_\circ,y_\circ) = r(t,x_\circ,y_\circ), y(t,x_\circ,y_\circ) = u(t,x_\circ,y_\circ)$

$\dot{x}(t,x_\circ,y_\circ) = \dot{r}(t,x_\circ,y_\circ), \dot{y}(t,x_\circ,y_\circ) = \dot{u}(t,x_\circ,y_\circ)$

that is each of $x(t,x_\circ,y_\circ), y(t,x_\circ,y_\circ)$ are unique solution for the system (1)

Section Two : The existence of the periodic solution for the system (1)

The problem of the existence of the periodic solution for the system (1) is connected with unique form with existence zero for all of the two functions

$\Delta_1^* = \Delta_1^*(x_\circ,y_\circ), \Delta_2^* = \Delta_2^*(x_\circ,y_\circ)$ in the form below :

$\Delta_1^* : D_f \times D_g \to R^n$,

$$\Delta_1^*(x_\circ,y_\circ) = \frac{1}{T}\int_0^T L\,(e^{A(t-s)}\,H_\infty(t,x_\circ,y_\circ))dt \quad , \qquad \text{..........(51)}$$

$\Delta_2^* : D_f \times D_g \to R^n$,

$$\Delta_2^*(x_\circ,y_\circ) = \frac{1}{T}\int_0^T L\,(e^{C(t-s)}\,Q_\infty(t,x_\circ,y_\circ))dt \quad , \qquad \text{..........(52)}$$

where $x_\infty(t,x_\circ,y_\circ)$ it is the end of the sequence $x_m(t,x_\circ,y_\circ)$ and $y_\infty(t,x_\circ,y_\circ)$ it is the end of the sequence $y_m(t,x_\circ,y_\circ)$. The existence of these two functions are impossible to prove except by successive approximation from the following two sequences of the functions

$\Delta^*_{1m} : D_f \times D_g \to R^n$,

$$\Delta^*_{1m}(x_\circ, y_\circ) = \frac{1}{T}\int_0^T L\,(e^{A(t-s)}\, H_m(t, x_\circ, y_\circ))dt \quad , \qquad \text{..........(53)}$$

$\Delta^*_{2m} : D_f \times D_f \to R^n$,

$$\Delta^*_{2m}(x_\circ, y_\circ) = \frac{1}{T}\int_0^T L\,(e^{C(t-s)}\, Q_m(t, x_\circ, y_\circ))dt \quad , \qquad \text{..........(54)}$$

where m=0,1,2,…….

Theorem 2:-

If were the assumptions and the conditions of the theorem 1 be given then the following inequalities :

$$\left\|\Delta^*_1(x_\circ, y_\circ) - \Delta^*_{1m}(x_\circ, y_\circ)\right\| \le$$

$$\le \left\langle \left((\frac{T}{2}Q(N_1+H)+QC_2) \;\; (\frac{T}{2}QH+QC_4) \;\; QC_3 \;\; QC_5 \right), \Omega_2^m (E-\Omega_2)^{-1} V_\circ \right\rangle = d^*_m$$

.........(55)

$$\left\|\Delta^*_2(x_\circ, y_\circ) - \Delta^*_{2m}(x_\circ, y_\circ)\right\| \le$$

$$\le \left\langle \left(R\delta_2 \;\; R\delta_4 \;\; (\frac{T}{2}R(N_2+J)+R\delta_3) \;\; (\frac{T}{2}RJ+R\delta_5) \right), \Omega_2^m (E-\Omega_2)^{-1} V_\circ \right\rangle = b^*_m$$

........(56)

satisfies for $m \ge 0, x_\circ \in D_f\,, y_\circ \in D_g$ and d^*_m, b^*_m are positive numerical constants.

and $\langle . \rangle$ denote to the non cross product in the Euclidean space R^n.

Proof :-

By the equations (51), (53) and the conditions (30) we have

$$\|\Delta_1^*(x_\circ,y_\circ)-\Delta_{1m}^*(x_\circ,y_\circ)\|\le$$
$$\le \beta_1(t)\|e^{A(t-s)}\|\,[\|A\|\|x_\infty(t,x_\circ,y_\circ)-x_m(t,x_\circ,y_\circ)\|+$$
$$+\|B(t)\|(\|x_\infty(t,x_\circ,y_\circ)-x_m(t,x_\circ,y_\circ)\|+\|\dot{x}_\infty(t,x_\circ,y_\circ)-\dot{x}_m(t,x_\circ,y_\circ)\|)]+$$
$$+\|e^{A(t-s)}\|\int_0^t[K_1(s)\|x_\infty(s,x_\circ,y_\circ)-x_m(s,x_\circ,y_\circ)\|+K_2(s)\|y_\infty(s,x_\circ,y_\circ)-y_m(s,x_\circ,y_\circ)\|+$$
$$+K_3(s)\|\dot{x}_\infty(s,x_\circ,y_\circ)-\dot{x}_m(t,x_\circ,y_\circ)\|+K_4(s)\|\dot{y}_\infty(s,x_\circ,y_\circ)-\dot{y}_m(s,x_\circ,y_\circ)\|]ds$$
$$\le\left\langle\left((\frac{T}{2}Q(N_1+H)+QC_2)\quad(\frac{T}{2}QH+QC_4)\quad QC_3\quad QC_5\right),\Omega_2^m(E-\Omega_2)^{-1}V_\circ\right\rangle=d_m^*$$

also by the equations (52), (54) and the conditions (30) we have

$$\|\Delta_2^*(x_\circ,y_\circ)-\Delta_{2m}^*(x_\circ,y_\circ)\|\le$$
$$\le\beta_2(t)\|e^{C(t-s)}\|\,[\|C\|\|y_\infty(t,x_\circ,y_\circ)-y_m(t,x_\circ,y_\circ)\|+$$
$$+\|D(s)\|(\|y_\infty(t,x_\circ,y_\circ)-y_m(t,x_\circ,y_\circ)\|+\|\dot{y}_\infty(t,x_\circ,y_\circ)-\dot{y}_m(t,x_\circ,y_\circ)\|)]+$$
$$+\|e^{C(t-s)}\|\int_0^t[L_1(s)\|x_\infty(s,x_\circ,y_\circ)-x_m(t,x_\circ,y_\circ)\|+L_2(s)\|y_\infty(s,x_\circ,y_\circ)-y_m(t,x_\circ,y_\circ)\|+$$
$$+L_3(s)\|\dot{x}_\infty(s,x_\circ,y_\circ)-\dot{x}_m(t,x_\circ,y_\circ)\|+L_4(s)\|\dot{y}_\infty(s,x_\circ,y_\circ)-\dot{y}_m(t,x_\circ,y_\circ)\|]ds$$

$$\le\left\langle\left(R\delta_2\quad R\delta_4\quad(\frac{T}{2}R(N_2+J)+R\delta_3)\quad(\frac{T}{2}RJ+R\delta_5)\right),\Omega_2^m(E-\Omega_2)^{-1}V_\circ\right\rangle=b_m^*$$

By the helping of the theorem 2 we introduce the following theorem and take into consideration the truth of the two inequalities (55) and (56)for $m\ge 0$.

Theorem 3:-

Let the right hand side from the system (1) are defined in the domain
$a\le x,\dot{x}\le b\,,c\le y,\dot{y}\le d\,,0\le t\le T$ on R^1 and assuming that the two sequences of the functions (53),(54) satisfies the following inequalities :

$$\left.\begin{array}{l} \min\limits_{\substack{a+\frac{T}{2}h\le x_\circ\le b-\frac{T}{2}h \\ c+\frac{T}{2}h^*\le y_\circ\le d-\frac{T}{2}h^*}} \Delta^*_{1m}(x_\circ,y_\circ)\le -d^*_m \\ \max\limits_{\substack{a+\frac{T}{2}h\le x_\circ\le b-\frac{T}{2}h \\ c+\frac{T}{2}h^*\le y_\circ\le d-\frac{T}{2}h^*}} \Delta^*_{1m}(x_\circ,y_\circ)\ge d^*_m \end{array}\right\} \quad \ldots\ldots..(57)$$

$$\left.\begin{array}{l} \min\limits_{\substack{a+\frac{T}{2}h\le x_\circ\le b-\frac{T}{2}h \\ c+\frac{T}{2}h^*\le y_\circ\le d-\frac{T}{2}h^*}} \Delta^*_{2m}(x_\circ,y_\circ)\le -b_m \\ \max\limits_{\substack{a+\frac{T}{2}h\le x_\circ\le b-\frac{T}{2}h \\ c+\frac{T}{2}h^*\le y_\circ\le d-\frac{T}{2}h^*}} \Delta^*_{2m}(x_\circ,y_\circ)\ge b_m \end{array}\right\} \quad \ldots\ldots..(58)$$

For all

$$d^*_m=\left\langle\left((\frac{T}{2}Q(N_1+H)+QC_2)\ \ (\frac{T}{2}QH+QC_4)\ \ QC_3\ \ QC_5\right),\Omega_2^m(E-\Omega_2)^{-1}V_\circ\right\rangle, m\ge 0$$

$$b^*_m=\left\langle\left(R\delta_2\ \ R\delta_4\ \ (\frac{T}{2}R(N_2+J)+R\delta_3)\ \ (\frac{T}{2}RJ+R\delta_5)\right),\Omega_2^m(E-\Omega_2)^{-1}V_\circ\right\rangle, m\ge 0$$

$$N^*=R^2J\sigma_\circ\ ,\ N=Q^2H\delta_\circ\ ,\ h^*=\frac{T}{2}N^*+R\delta_1\ ,\ h=\frac{T}{2}N+QC_1$$

then the system (1) have periodic solutions
$x=x(t,x_\circ,y_\circ), y=y(t,x_\circ,y_\circ)$
for $x_\circ\in[a+\frac{T}{2}h,b-\frac{T}{2}h]\ ,y_\circ\in[c+\frac{T}{2}h^*,d-\frac{T}{2}h^*]$.

Proof :-

Let x_1,x_2 be any two points in the interval $[a+\frac{T}{2}h,b-\frac{T}{2}h]$, y_2 , y_1 be any two points in the interval $[c+\frac{T}{2}h^*,d-\frac{T}{2}h^*]$ so that

$$\left.\begin{aligned}\Delta^*_{1m}(x_1,y_1) &= \min_{\substack{a+\frac{T}{2}h\le x_\circ\le b-\frac{T}{2}h\\ c+\frac{T}{2}h^*\le y_\circ\le d-\frac{T}{2}h^*}} \Delta^*_{1m}(x_\circ,y_\circ)\\ \Delta^*_{1m}(x_2,y_2) &= \max_{\substack{a+\frac{T}{2}h\le x_\circ\le b-\frac{T}{2}h\\ c+\frac{T}{2}h^*\le y_\circ\le d-\frac{T}{2}h^*}} \Delta^*_{1m}(x_\circ,y_\circ)\end{aligned}\right\} \quad ....(59)$$

$$\left.\begin{aligned}\Delta^*_{2m}(x_1,y_1) &= \min_{\substack{a+\frac{T}{2}h\le x_\circ\le b-\frac{T}{2}h\\ c+\frac{T}{2}h^*\le y_\circ\le d-\frac{T}{2}h^*}} \Delta^*_{2m}(x_\circ,y_\circ)\\ \Delta^*_{2m}(x_2,y_2) &= \max_{\substack{a+\frac{T}{2}h\le x_\circ\le b-\frac{T}{2}h\\ c+\frac{T}{2}h^*\le y_\circ\le d-\frac{T}{2}h^*}} \Delta^*_{2m}(x_\circ,y_\circ)\end{aligned}\right\} \quad ....(60)$$

by using the inequalities (55),(56),(57),(58) we have

$$\left.\begin{aligned}\Delta^*_1(x_1,y_1) &= \Delta^*_{1m}(x_1,y_1)+(\Delta^*_1(x_1,y_1)-\Delta^*_{1m}(x_1,y_1))<0\\ \Delta^*_1(x_2,y_2) &= \Delta^*_{1m}(x_2,y_2)+(\Delta^*_1(x_2,y_2)-\Delta^*_{1m}(x_2,y_2))>0\end{aligned}\right\} \quad ....(61)$$

$$\left.\begin{aligned}\Delta^*_2(x_1,y_1) &= \Delta^*_{2m}(x_1,y_1)+(\Delta^*_2(x_1,y_1)-\Delta^*_{2m}(x_1,y_1))<0\\ \Delta^*_2(x_2,y_2) &= \Delta^*_{2m}(x_2,y_2)+(\Delta^*_2(x_2,y_2)-\Delta^*_{2m}(x_2,y_2))>0\end{aligned}\right\} \quad ....(62)$$

and from the continuity of the two functions $\Delta^*_1(x_\circ,y_\circ),\Delta^*_2(x_\circ,y_\circ)$ and the two inequalities (61),(62) then there exist an isolated single point $(x_\infty,y_\infty)=(x_\circ,y_\circ)$ and

$x_\infty\in[x_1,x_2], y_\infty\in[y_1,y_2]$ where $\Delta^*_1(x_\circ,y_\circ)=0, \Delta^*_2(x_\circ,y_\circ)=0$

this mean that for the system (1) periodic solution

$x=(t,x_\circ,y_\circ), y=(t,x_\circ,y_\circ)$

for $x_\circ\in[a+\frac{T}{2}h, b-\frac{T}{2}h]$, $y_\circ\in[c+\frac{T}{2}h^*, d-\frac{T}{2}h^*]$.

Remark 1 :-

We completed the formulation for the text of the theorem 3 with the proof when $R^n=R^1$ that is when $x_\circ, y_\circ$ are two non vectors quantities

Theorem 4 :-

Let

$\Delta_1^* : D_f \times D_g \to R^n$

$$\Delta_1^*(x_\circ, y_\circ) = \frac{1}{T}\int_0^T L\,(e^{A(t-s)}\,H_\infty(t, x_\circ, y_\circ))dt \qquad \text{..........(63)}$$

$\Delta_2^* : D_f \times D_g \to R^n$

$$\Delta_2^*(x_\circ, y_\circ) = \frac{1}{T}\int_0^T L\,(e^{C(t-s)}\,Q_\infty(t, x_\circ, y_\circ))dt \qquad \text{..........(64)}$$

where $x_\infty(t, x_\circ, y_\circ)$ it is the end of the periodic sequence (25) and the function
$y_\infty(t, x_\circ, y_\circ)$ it is the end of the periodic sequence (26) then the following inequalities

$$\left\|\Delta_1^*(x_\circ, y_\circ)\right\| \le M_{13} \qquad \text{.........(65)}$$

where

$$M_{13} = \frac{T}{2}Q(N_1 + H)M_9[\delta_\circ Q + \frac{T^2}{4}Q^2 HC_1 M_3 + \frac{T}{2}QC_1 + \frac{T}{2}Q\,HM_3\delta_\circ Q + \frac{T^3}{8}Q^3 H^2 M_3^2 C_1 + \frac{T^2}{4}Q^2 H\,C_1 M_3] + QC_1(\frac{T}{2}QHM_3 + 1)$$

$$M_3 = (1 - \frac{T}{2}Q\,H)^{-1},\ M_9 = (1 - \frac{T^2}{4}Q(N_1 + H)(1 + \frac{T}{2}Q\,H\,M_3))^{-1}$$

$$\left\|\Delta_2^*(x_\circ, y_\circ)\right\| \le M_{14} \qquad \text{.......(66)}$$

where

$$M_{14} = \frac{T}{2}R(N_2 + J)M_{11}[\sigma_\circ R + \frac{T^2}{4}R^2 J\,\delta_1 M_5 + \frac{T}{2}R\delta_1 + \frac{T}{2}R\,J\,M_5\,\sigma_\circ R + \frac{T^3}{8}R^3 J^2\,M_5^2\delta_1 + \frac{T^2}{4}R^2 J\,M_5\delta_1] + R\delta_1(\frac{T}{2}R\,J\,M_5 + 1)$$

$$M_5 = (1 - \frac{T}{2}R\,J)^{-1}\ ,\ M_{11} = (1 - \frac{T^2}{4}R(N_2 + J)(1 + \frac{T}{2}R\,J\,M_5))^{-1}$$

$$\left\|\Delta_1^*(x_\circ^1,y_\circ^1)-\Delta_1^*(x_\circ^2,y_\circ^2)\right\| \le [\frac{T}{2}E_5W_1W_3W_5W_6\,W_9W_{10}W_{11}\gamma_1+W_1W_5\,W_8\gamma_2\,]\left\|x_\circ^1-x_\circ^2\right\|Q+$$

$$+[W_3W_6\,W_9W_{10}W_{11}\gamma_1]\left\|y_\circ^1-y_\circ^2\right\|R \qquad ....(67)$$

$$\left\|\Delta_2^*(x_\circ^1,y_\circ^1)-\Delta_2^*(x_\circ^2,y_\circ^2)\right\| \le [\frac{T}{2}E_5W_1W_3W_5W_6\,W_9W_{10}W_{11}\gamma_3+W_1W_5\,W_8\gamma_4]\left\|x_\circ^1-x_\circ^2\right\|Q+$$

$$+[W_3W_6\,W_9W_{10}W_{11}\gamma_3\,]\left\|y_\circ^1-y_\circ^2\right\|R \qquad ....(68)$$

satisfies for $x_\circ, x_\circ^1, x_\circ^2 \in D_f$, $y_\circ, y_\circ^1, y_\circ^2 \in D_g$

Where

$$E_1=\frac{T}{2}Q(N_1+H)+QC_2\,,E_2=\frac{T}{2}QH+QC_4\,,E_3=QC_3\,,E_4=QC_5$$

$$E_5=R\delta_2\,,E_6=R\delta_4\,,E_7=\frac{T}{2}R(N_2+J)+R\delta_3\,\,,E_8=\frac{T}{2}RJ+R\delta_5$$

$$\gamma_1=E_1W_5W_8a_1+E_1E_2W_2W_5W_8a_1+E_2E_3W_2+E_2E_4W_2W_5W_7W_8a_1a_4+$$
$$+E_2E_4W_2W_7a_5+E_3+E_4W_5W_7W_8a_1a_4+E_4W_7a_5$$

$$\gamma_2=E_1W_5W_8W_{10}W_{11}a_1a_7+E_1+E_1E_2W_2W_5W_8W_{10}W_{11}a_1a_7+E_1E_2W_2+$$
$$+E_2E_3W_2W_{10}W_{11}a_7+E_2E_4W_2\,W_5W_7W_8W_{10}W_{11}a_1a_4a_7+E_2E_4W_2\,W_7a_4+$$
$$+E_2E_4W_2\,W_7W_{10}W_{11}a_5a_7+E_3W_{10}W_{11}a_7+E_4W_5W_7W_8W_{10}W_{11}a_1a_4a_7+$$
$$+E_4W_7a_4+E_4W_7W_{10}W_{11}a_5a_7$$

$$\gamma_3=E_5W_5W_8a_1+E_1E_6W_2W_5W_8a_1+E_6E_3W_2+E_4E_6W_2W_5W_7W_8a_1a_4+$$
$$+E_4E_6W_2W_7a_5+E_7+E_8W_5W_7W_8a_1a_4+E_8W_7\,a_5$$

$$\gamma_4=E_5W_5W_8W_{10}W_{11}a_1a_7+E_5+E_1E_6W_2W_5W_8W_{10}W_{11}a_1a_7+E_1E_6W_2+$$
$$+E_3E_6W_2W_{10}W_{11}a_7+E_4E_6W_2W_5W_7W_8W_{10}W_{11}a_1a_4a_7+E_4E_6W_2W_7a_4+$$
$$+E_4E_6W_2W_7W_{10}W_{11}a_5a_7+E_7W_{10}W_{11}a_7+E_8W_5W_7W_8W_{10}W_{11}a_1a_4a_7+$$
$$+E_8W_7a_4+E_8W_7W_{10}W_{11}a_5a_7$$

$$W_1=(1-\frac{T}{2}E_1)^{-1}\,,W_2=(1-E_2)^{-1}\,,W_3=(1-\frac{T}{2}E_7)^{-1}\,,W_4=(1-E_8)^{-1}$$

$a_1 = [\frac{T}{2} E_2 E_3 W_1 W_2 + \frac{T}{2} E_3 W_1]$, $W_5 = (1 - \frac{T}{2} E_1 E_2 W_1 W_2)^{-1}$

$a_2 = [\frac{T}{2} E_2 E_4 W_1 W_2 + \frac{T}{2} E_4 W_1]$, $W_6 = (1 - \frac{T}{2} E_5 W_3 W_5 a_1)^{-1}$

$a_3 = [\frac{T}{2} E_5 W_3 W_5 a_2 + \frac{T}{2} E_8 W_3]$, $W_7 = (1 - E_4 E_6 W_2 W_4)^{-1}$

$a_4 = [E_5 W_4 + E_1 E_6 W_2 W_4]$, $W_8 = (1 - W_5 W_7 a_2 a_4)^{-1}$

$a_5 = [E_3 E_6 W_2 W_4 + E_7 W_4]$, $W_9 = (1 - \frac{T}{2} E_3 E_6 W_2 W_3 E_6)^{-1}$

$a_6 = [\frac{T}{2} E_4 E_6 W_2 W_3 W_6 + a_3 W_6]$, $W_{10} = (1 - W_7 W_9 a_6 a_5)^{-1}$

$a_7 = [\frac{T}{2} E_1 E_6 W_2 W_3 W_6 W_9 + W_7 W_9 a_6 a_4]$, $W_{11} = (1 - W_5 W_8 W_{10} a_1 a_7)^{-1}$

Proof :-

From properties of the function $x_\infty(t, x_\circ, y_\circ)$ and the function $y_\infty(t, x_\circ, y_\circ)$ that fixative by theorem 1 then each of the two functions $\Delta_1^*(x_\circ, y_\circ), \Delta_2^*(x_\circ, y_\circ)$,
$x_\circ \in D_f, y_\circ \in D_g$ continuous and bounded by non negative constants M_{13}, M_{14} on arrangement in the domain (2).
from the relation (63) we find that

$$\|\Delta_1^*(x_\circ, y_\circ)\| \le \|L(e^{A(t-s)} H_\infty(t,, x_\circ, y_\circ))\|$$

and by using lemma 1 we get

$$\|\Delta_1^*(x_\circ, y_\circ)\| \le \frac{T}{2} Q(N_1 + H)\|x_\infty(t, x_\circ, y_\circ)\| + \frac{T}{2} QH\|\dot{x}_\infty(t, x_\circ, y_\circ)\| + QC_1 \quad .....(69)$$

since that $x_\infty(t, x_\circ, y_\circ)$ satisfy the integral equation (28) we find that

$$\|x_\infty(t, x_\circ, y_\circ)\| \le \delta_\circ Q + \frac{T^2}{4} Q(N_1 + H)\|x_\infty(t, x_\circ, y_\circ)\| + \frac{T^2}{4} QH\|\dot{x}_\infty(t, x_\circ, y_\circ)\| + \frac{T}{2} QC_1$$

......(70)

also we have

$$\|\dot{x}_\infty(t,x_\circ,y_\circ)\| \le \frac{T}{2}Q(N_1+H)M_3\|x_\infty(t,x_\circ,y_\circ)\| + QC_1M_3 \qquad \text{........(71)}$$

and by substitutions the inequality (71) in the inequality (70) we obtain

$$\|x_\infty(t,x_\circ,y_\circ)\| \le \delta_\circ QM_9 + \frac{T}{2}QC_1(\frac{T}{2}QHM_3+1)M_9 \qquad \text{..........(72)}$$

and by substitutions the inequality (72) in the inequality (71) we obtain

$$\|\dot{x}_\infty(t,x_\circ,y_\circ)\| \le \frac{T}{2}Q(N_1+H)M_3M_9\delta_\circ Q + \frac{T^2}{4}Q^2(N_1+H)C_1M_3M_9(\frac{T}{2}QHM_3+1) + \\ +QC_1M_3 \qquad \text{........(73)}$$

by substitutions the two inequalities (72),(73) in the inequality (70) we obtain (65).
From the relation (64) we find that

$$\|\Delta_2^*(x_\circ,y_\circ)\| \le \left\| L(e^{C(t-s)}Q_\infty(t,,x_\circ,y_\circ)) \right\|$$

and by using lemma 1 we get

$$\|\Delta_2^*(x_\circ,y_\circ)\| \le \frac{T}{2}R(N_2+J)\|y_\infty(t,x_\circ,y_\circ)\| + \frac{T}{2}RJ\|\dot{y}_\infty(t,x_\circ,y_\circ)\| + R\delta_1 \qquad \text{..........(74)}$$

since that $y_\infty(t,x_\circ,y_\circ)$ satisfy the integral equation (29) we find

$$\|y_\infty(t,x_\circ,y_\circ)\| \le \sigma_\circ R + \frac{T^2}{4}R(N_2+J)\|y_\infty(t,x_\circ,y_\circ)\| + \frac{T^2}{4}RJ\|\dot{y}_\infty(t,x_\circ,y_\circ)\| + \frac{T}{2}R\delta_1 \qquad \text{.........(75)}$$

also we have

$$\|\dot{y}_\infty(t,x_\circ,y_\circ)\| \le \frac{T}{2}R(N_2+J)M_5\|y_\infty(t,x_\circ,y_\circ)\| + R\delta_1M_5 \qquad \text{..........(76)}$$

and by substitutions the inequality (76) in the inequality (75) we obtain

$$\|y_\infty(t,x_\circ,y_\circ)\| \le \sigma_\circ RM_{11} + \frac{T}{2}R\delta_1(\frac{T}{2}RJM_5+1)M_{11} \qquad \text{.........(77)}$$

and by substitutions the inequality (77) in the inequality (76) we obtain

$$\|\dot{y}_\infty(t,x_\circ,y_\circ)\| \le \frac{T}{2}R(N_2+J)M_5M_{11}\sigma_\circ R + \frac{T^2}{4}R^2(N_2+J)\delta_1M_5M_{11}(\frac{T}{2}RJM_5+1) + \\ +R\delta_1M_5 \qquad \text{........(78)}$$

by substitutions the two inequalities (77),(78) in the inequality (74) we obtain (66).
from the relation (63) we find that

$$\|\Delta_1^*(x_\circ^1,y_\circ^1)-\Delta_1^*(x_\circ^2,y_\circ^2)\|\le E_1\|x_\infty(t,x_\circ^1,y_\circ^1)-x_\infty(t,x_\circ^2,y_\circ^2)\|+$$
$$+E_2\|\dot{x}_\infty(t,x_\circ^1,y_\circ^1)-\dot{x}_\infty(t,x_\circ^2,y_\circ^2)\|+E_3\|y_\infty(t,x_\circ^1,y_\circ^1)-y_\infty(t,x_\circ^2,y_\circ^2)\|+$$
$$+E_4\|\dot{y}_\infty(t,x_\circ^1,y_\circ^1)-\dot{y}_\infty(t,x_\circ^2,y_\circ^2)\| \qquad ....(79)$$

where

$$x(t,x_\circ^k,y_\circ^k)=x_\circ^k e^{At}+L^2\,(e^{A(t-s)}H(t,x_\circ^k,y_\circ^k)) \qquad ....(80)$$
$$y(t,x_\circ^k,y_\circ^k)=y_\circ^k e^{Ct}+L^2\,(e^{C(t-s)}\,Q(t,x_\circ^k,y_\circ^k)) \qquad ....(81)$$

where k =1,2.

since $x_\infty(t,x_\circ,y_\circ),y_\infty(t,x_\circ,y_\circ)$ satisfies the two equations (28),(29) on arrangement .

from the relation (80) we find that

$$\|x_\infty(t,x_\circ^1,y_\circ^1)-x_\infty(t,x_\circ^2,y_\circ^2)\|\le W_1\|x_\circ^1-x_\circ^2\|\,Q+\frac{T}{2}E_2W_1\|\dot{x}_\infty(t,x_\circ^1,y_\circ^1)-\dot{x}_\infty(t,x_\circ^2,y_\circ^2)\|$$
$$+\frac{T}{2}E_3W_1\|y_\infty(t,x_\circ^1,y_\circ^1)-y_\infty(t,x_\circ^2,y_\circ^2)\|+\frac{T}{2}E_4W_1\|\dot{y}_\infty(t,x_\circ^1,y_\circ^1)-\dot{y}_\infty(t,x_\circ^2,y_\circ^2)\| \qquad ....(82)$$

also we find that

$$\|\dot{x}_\infty(t,x_\circ^1,y_\circ^1)-\dot{x}_\infty(t,x_\circ^2,y_\circ^2)\|\le E_1W_2\|x_\infty(t,x_\circ^1,y_\circ^1)-x_\infty(t,x_\circ^2,y_\circ^2)\|$$
$$+E_3W_2\|y_\infty(t,x_\circ^1,y_\circ^1)-y_\infty(t,x_\circ^2,y_\circ^2)\|+E_4W_2\|\dot{y}_\infty(t,x_\circ^1,y_\circ^1)-\dot{y}_\infty(t,x_\circ^2,y_\circ^2)\|$$
$$...........(83)$$

from the relation (81) we find that

$$\|y_\infty(t,x_\circ^1,y_\circ^1)-y_\infty(t,x_\circ^2,y_\circ^2)\|\le W_3\|y_\circ^1-y_\circ^2\|R+\frac{T}{2}E_5W_3\|x_\infty(t,x_\circ^1,y_\circ^1)-x_\infty(t,x_\circ^2,y_\circ^2)\|$$
$$+\frac{T}{2}E_6W_3\|\dot{x}_\infty(t,x_\circ^1,y_\circ^1)-\dot{x}_\infty(t,x_\circ^2,y_\circ^2)\|+\frac{T}{2}E_8W_3\|\dot{y}_\infty(t,x_\circ^1,y_\circ^1)-\dot{y}_\infty(t,x_\circ^2,y_\circ^2)\|$$
$$..........(84)$$

also we find that

$$\|\dot{y}_\infty(t,x_\circ^1,y_\circ^1)-\dot{y}_\infty(t,x_\circ^2,y_\circ^2)\|\le E_5W_4\|x_\infty(t,x_\circ^1,y_\circ^1)-x_\infty(t,x_\circ^2,y_\circ^2)\|+$$
$$+E_6W_4\|\dot{x}_\infty(t,x_\circ^1,y_\circ^1)-\dot{x}_\infty(t,x_\circ^2,y_\circ^2)\|+E_7W_4\|y_\infty(t,x_\circ^1,y_\circ^1)-y_\infty(t,x_\circ^2,y_\circ^2)\|$$
$$...........(85)$$

after chain from substitutions in the inequalities (82),(83),(84),(85)we find

$$\|x_\infty(t,x_\circ^1,y_\circ^1)-x_\infty(t,x_\circ^2,y_\circ^2)\|\le[W_1W_5W_8+\frac{T}{2}E_5W_1W_3(W_5)^2W_6W_8W_9W_{10}W_{11}\,a_1+$$
$$+W_1(W_5)^2(W_8)^2W_{10}W_{11}a_1a_7]\|x_\circ^1-x_\circ^2\|Q+W_3W_5W_6\,W_8W_9W_{10}W_{11}\,a_1\|y_\circ^1-y_\circ^2\|R \quad ....(86)$$

$$\|y_\infty(t,x_\circ^1,y_\circ^1)-y_\infty(t,x_\circ^2,y_\circ^2)\|\le[\frac{T}{2}E_5W_1W_3W_5W_6W_9W_{10}W_{11}+$$
$$+W_1W_5W_8W_{10}W_{11}a_7]\|x_\circ^1-x_\circ^2\|Q+W_3W_6\,W_9W_{10}W_{11}\|y_\circ^1-y_\circ^2\|R \quad ....(87)$$

$$\left\|\dot y_\infty(t,x_\circ^1,y_\circ^1)-\dot y_\infty(t,x_\circ^2,y_\circ^2)\right\|\le[\frac{T}{2}E_5W_1W_3(W_5)^2W_6\,W_7W_8W_9W_{10}W_{11}a_1a_4+$$
$$+W_1(W_5)^2W_7(W_8)^2W_{10}W_{11}a_1a_4a_7+W_1\,W_5\,W_7W_8a_4+$$
$$+\frac{T}{2}E_5W_1W_3W_5W_6\,W_7W_9W_{10}W_{11}a_5+W_1W_5W_7W_8W_{10}W_{11}a_5a_7]\|x_\circ^1-x_\circ^2\|Q+$$
$$+[W_3\,W_5W_6W_7W_8W_9W_{10}W_{11}a_1a_4+W_3W_6W_7W_9W_{10}W_{11}a_5]\|y_\circ^1-y_\circ^2\|R \quad ....(88)$$

$$\|\dot x_\infty(t,x_\circ^1,y_\circ^1)-\dot x_\infty(t,x_\circ^2,y_\circ^2)\|\le\frac{T}{2}E_1E_5W_1W_2W_3(W_5)^2W_6W_8W_9W_{10}W_{11}a_1+$$
$$+E_1W_1W_2(W_5)^2(W_8)^2W_{10}W_{11}a_1a_7+E_1W_1W_2W_5W_8+$$
$$+\frac{T}{2}E_3E_5W_1W_2W_3W_5W_6W_9W_{10}W_{11}+E_3W_1W_2W_5W_8W_{10}W_{11}a_7+$$
$$+\frac{T}{2}E_4E_5W_1W_2W_3(W_5)^2W_6W_7W_8W_9W_{10}W_{11}a_1a_4+$$
$$+E_4W_1W_2(W_5)^2W_7(W_8)^2W_{10}W_{11}a_1a_4a_7+E_4W_1W_2W_5W_7W_8a_4+$$
$$+\frac{T}{2}E_4E_5W_1W_2W_3W_5W_6W_7W_9W_{10}W_{11}a_5+E_4W_1W_2W_5W_7W_8W_{10}W_{11}a_5a_7]\|x_\circ^1-x_\circ^2\|Q+$$
$$+[E_1\,W_2W_3W_5W_6W_8W_9W_{10}W_{11}a_1+E_3W_2W_3W_6W_9W_{10}W_{11}+$$
$$+E_4W_2W_3W_5W_6W_7W_8W_9W_{10}W_{11}a_1a_4+E_4W_2W_3W_6W_7W_9W_{10}W_{11}a_5]\|y_\circ^1-y_\circ^2\|R \quad ....(89)$$

by substitutions the inequalities (86),(87),(88),(89) in the inequality (79) we obtain (67).
also by the relation (64) we find that

$$\|\Delta_2^*(x_\circ^1, y_\circ^1) - \Delta_2^*(x_\circ^2, y_\circ^2)\| \le E_5\|x_\infty(t, x_\circ^1, y_\circ^1) - x_\infty(t, x_\circ^2, y_\circ^2)\| +$$
$$+E_6\|\dot{x}_\infty(t, x_\circ^1, y_\circ^1) - \dot{x}_\infty(t, x_\circ^2, y_\circ^2)\| + E_7\|y_\infty(t, x_\circ^1, y_\circ^1) - y_\infty(t, x_\circ^2, y_\circ^2)\| +$$
$$+E_8\|\dot{y}_\infty(t, x_\circ^1, y_\circ^1) - \dot{y}_\infty(t, x_\circ^2, y_\circ^2)\| \quad ....(90)$$

by substitutions the inequalities (86),(87),(88),(89) in the inequality (90) we obtain (68).

Remark 2:-

The theorem 4 confirm the stability of the solution for the system of non linear differential equations that is when a slight change happening in the point $(x_\circ, y_\circ)$ then a slight change will happen in the two functions
$\Delta_1^* = \Delta_1^*(x_\circ, y_\circ), \Delta_2^* = \Delta_2^*(x_\circ, y_\circ)$
[for this remark return to [3]].

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