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Classification of some arcs of type (m,n) in PG(2,19)

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الخلاصة:

تضمن هذا البحث دراسة لأوزان نقاط ومستقيمات الأقواس -(k,n) ابتداء من $L_0 = 1$ ولغاية $L_0 = 381$ في المستوي PG(2,19) والتي كانت غير معروفة إلى حد الآن، وبالنظر لكبر عدد هذه النقاط والمستقيمات فقد تم تقسيم البحث الى ثلاث مر احل حيث تضمنت المرحلة الأولى اختبار جميع القيم من $L_0 = 1$ ولغاية $L_0 = 171$ حيث توصلنا عند القيمة $L_0 = 171$ إلى القوس-(210,39; f) من النوع (20,39) أما بقية القيم من $L_0 = 1$ إلى $L_0 = 170$ فلا تحقق هذه الخاصية. ثم اخترنا في المرحلة الثانية دراسة هذه الأقواس من $L_0 = 340$ وحتى $L_0 = 381$ وتمكنا في هذه المرحلة من إيجاد نوعين آخرين من الأقواس عندما $L_0 = 342$ وأخر عندما $L_0 = 361$ وهي على التوالي القوس -(39,21;f) من النوع (2,21) والقوس -(20,20;f) من النوع (1,20)، ثم قمنا ببراهين للنظريات الخاصة بالحالات أعلاه.

وتضمنت المرحلة الأخيرة برهان عدم وجود القوس -(k,n;f) من النوع (n-19,n) في هذا المستوي لجميع قيم L_0 المحصورة بين 362 إلى 381.

Abstract:

This paper conclude the study of the weights of points and lines of (k,n)-arcs from $L_0 = 1$ until $L_0 = 381$, since the number of points and lines of this plane is very large, so we divide this search in to three parts.

The first part contained the existence of these arcs from $L_0 = 1$ to $L_0 = 171$ and we show that when $L_0 = 171$ the (210,39;f)-arc of type (20,39) exist and all the other cases from $L_0 = 1$ to $L_0 = 170$ does not exist.

In the second part we studied these arcs from $L_0 = 340$ to $L_0 = 381$ and we saw that there are two types of these arcs when $L_0 = 342$ and $L_0 = 361$ which are respectively (39,21;f)- arc of type (2,21) and (20,20;f)-arc of type (1,20), then proved the theorem of these special cases are given.

Finally in the last step we prove that (k,n;f)- arc of type (n-19,n) does not exist in PG(2,19) for all values of L_0 from 362 to 381.

1. Introduction

Let GF(q) denote the Galois field of q elements. A projective plane π over GF(q) is a two–dimensional projective space contains q^2 +q +1 points and q^2 +q +1 lines, with q+1 points on a line and q+1 lines on each point. Any two points have exactly one line joining them and any two lines meet in just one point.

A (k,n)- arc K in a finite projective plane of order q is defined as a set of k points satisfying the following two conditions:

i) no line of the plane contains more than n points of K.

ii) there exist at least one line of the plane which contains n points of K.

In the papers of D` Agostini, E. "Sulla caratterizzazione celle (k,n;f)-calotte di tipo (n-2,n)" [1] and [2],[3],[4],[5],[6],[7] an account has been given of some results concerning weighted (k,n)- arcs in finite and particularly Galois planes for q=3,5,7,9,13 and 17. These objects are also called (k,n;f)- arcs. In this paper we are concerned with extending the work in the projective plane of order 19.

The definition of a weighted arc given in previous papers. Thus we are concerned with a set K of k>0 points in PG(2,q) to each point p of which is assigned a natural no. f(p) called its weight and such that the total weight of the points on any line does not exceed a given natural no. n, line having total weight i is called an i- secant of K. Points not included in K are assigned the weight zero, and use L_j to denote number of points of weight j for j=0,1,…,w. Let t_i denote the number of i- secants of K for which i=0,1,…,n. If the values of i for which t_i is non-zero are $m_1 < .... < n$ then K is said to be of type $(m_1, m_2, ..., n)$.

(1-1) Definition

If K is a (k,n;f)- arc, we call the type of K the set Imf.

(1-2) Definition

If K is a (k,n;f)- arc of type (0,n) then Imf={0,w} and we get the ordinary case, i.e, a (k,n;f)- arc is a (k,n)- arc.

(1-3) Lemma

The weight W of a (k,n;f)- arc of type (m,n) satisfies $m(q+1) \le W \le (n-w)(q+1)+w$

Proof: See [3] (p.11).

(1-4) Theorem

Let K be a (k,n;f)- arc of type (m,n), m>0 and let V_m^s and V_n^s be respectively the number of lines of weight m and number of lines of weight n passing through a point of weight s, then

$$(n-m)V_m^s = (n-s)(q+1)-(W-s)$$

$$(n-m)V_n^s = (W-s)-(m-s)(q+1)$$

Proof: see [3] (p.12).

(1-5) Lemma

A necessary condition for the existence of a (k,n;f)- arc K of type (m,n), m > 0 is that q ≡0 mod (n-m).

Proof: see [3] (p.14).

Hint:

For A(k,n;f)- arc of type (1,n), it is necessary that Imf={0,1,w} where w≥2 and q ≡ 0 mod (n-1).

From theorem (1-4), we obtain the following:

$$V_1^0 = \frac{qw}{n-1} \qquad V_n^0 = q - \frac{qw}{(n-1)} + 1$$

$$V_1^1 = \frac{q(w-1)}{n-1} \qquad V_n^1 = \frac{q(n-w)}{(n-1)} + 1$$

$$V_1^w = 0 \qquad V_n^w = q+1$$

$$t_n = \frac{q}{(n-1)}[(n-w-1)(q+1)+n]+1$$

$$t_1 = \frac{q}{(n-1)}(qw+w-n)$$

(1-6) Theorem

If K is a (k,n;f)- arc of type (m,n) then its necessary that $w \le n-m$.

Proof: see [3] (p.14).

2. (k,n,f)- arcs of type (m,n)

We now study (k,n,f)- arcs of type (m,n), $m = n-19$. For this it is necessary that n≥19.

If n = 19 we get to consider a (k,19)- arc having only 0-secant and 19-secant.

Hence from now we must assume $n \ge 20$ from theorem (1-6) we must have $w \le 19$.

Hence there is no points of weight greater than 19, that is, $L_j = 0$, j >19.

Apply lemma (1-3), then the maximal and minimal weight of the K, when $m = n-19$ and $w = 19$ is:

$(n-19)(q+1) \le W \le (n-19)(q+1)+19$

3. (k,n;f)-arcs of type (n-19,n) with Lj>0 for j=0,1,2, Lj =0,j=3,4,…,19 in PG(2,19)

Let t_{n-19} be the number of lines of weight $n-19$ and t_n the number of the lines of weight n; then

$t_{n-19} + t_n = q^2 + q + 1$ .... (3-1)

$(n-19)t_{n-19} + nt_n = W(q+1) = (n-19)(q+1)^2$ .... (3-2)

Multiplying equation (3-1) by (n-19) and subtracting from equation (3-2), we get

$t_n = \frac{1}{19}(n-19)q$ ....(3-3)

substitute the value of t_n in equation (3-1), we have

$t_{n-19} = \frac{1}{19}(19q^2 + 38q - nq + 19)$ .... (3-4)

Let N be an n-secant which has no points of weight 0 and suppose that on N there are α points of weight 1 and β points of weight 2,counting points of N gives $\alpha + \beta$ =q+1

And weight of points on N,we get α +2 β =n

So α =2(q+1) –n ....(3-5)

β =n–(q+1) ....(3-6)

Counting incidence between points of weight 1 and n-secant gives

$L_1 V_n^1 = t_n \alpha$

By using theorem (1-4) and equations (3-3) & (3-5), we get

$L_1 = (n-19)(2q+2-n)$ .... (3-7)

Similarly by using theorem (1-4) and equations (3-3) & (3-6), we have

$L_2 = \frac{(n-19)(n-q-1)}{2}$ ....(3-8)

Since the points in the plane

$L_0 + L_1 + L_2 = q^2 + q + 1$

$L_0 = q^2 + q + 1 - L_1 - L_2$ i.e

then by equations (3-7)&(3-8), we obtained

$L_0 = (q^2 + q + 1) - (n-19)(2q+2-n) - \frac{(n-19)(n-q-1)}{2}$ .... (3-9)

Hence, we get

$2q^2 + (59-3n)q + n^2 - 22n + 59 - 2L_0 = 0$ .... (3-10)

The solution of (3-10) exists w.r.t q if

$(59-3n)^2 - 8(n^2 - 22n + 59 - 2L_0)$ is square,

i.e $\delta^2 = (n-89)^2 - (4912-16L_0)$ ……………………………………(3-11)
From equation (3-11) it follows that a necessary condition for the existence of such a (k,n,f)-arc of type (n-5,n), for which $L_j = 0$, $j = 3,4,.....,19$ is that
$(n-89)^2 - (4912-16L_0)$ should be a square.

4. Existence of (k,n;f)-arcs for L0 =1,2,…,171

In this section we will discuss the (k,n;f)- arcs when $L_0 = 1,2,.....,171$ of type (n-19,n) in PG(2,19).

(4-1) The cases for L_0 =1,…,170

<u>First case</u>: Let $L_0 = 1$, the equation (3-10) becomes:
$2q^2 + (59-3n)q + n^2 - 22n + 57 = 0$ ……………………………………(4-1-1)
The solution of (4-1) exists w.r.t q if
$(59-3n)^2 - 8(n^2 - 22n + 57)$ is square, i.e
$\delta^2 = (n-89)^2 - (4912-16L_0)$ ……………………………………(4-1-2)
So we can get
$\delta^2 = (n-89)^2 - 4896$
$(\delta-(n-89))(\delta+(n-89)) = -4896$

The possibilities of the factors of (-4896) are given bellow in (table-1). We only need to consider possibilities in which the factors have the same parity because
$(\delta-(n-89)) + (\delta+(n-89)) = 2\delta$

Also we need to list those values for which γ and n are non-negative integers.
Note that q is obtained from equation (4-1-1)

(Table -1)

When t_0 =1

δ-(n-89)	δ +(n-89)	δ	n	q
-2	2448	1223	1314	665, 1276.5
-4	1224	610	703	360, 665
-8	612	302	399	209, 360
-16	306	145	250	209, 136.5
-68	72	2	159	104, 105
72	-68	2	19	0 , 1
-18	272	127	234	385, 129
-24	204	90	203	160, 115
102	-48	27	14	-11, 2.5
-48	102	27	164	115, 101.5
136	-36	50	3	0, -50
-36	136	50	175	129, 104
144	-34	55	0	-1, -28.5
-34	144	55	178	132.5, 105
-6	816	405	500	259, 923
-12	408	198	299	259, 320

We have checked all of these compatible with n and δ being non – negative integers and with q $\equiv 0$ mod (n-m) and we find there does not exist a (k,n;f) – arc of type (n-19,n) when L_o =1.

Second case: For the value of L_o =2

The equation (3-10) becomes

$2q^2 + (59 - 3n)q + n^2 - 22n + 55 = 0$

By equation (4-1-2) we get

$\delta^2 = (n-89)^2 - 4880$

Whence $(\delta - (n-89))\,(\delta + (n-89)) = -4880$

The possibilities of the factors of (–4880) are given in table-2, we only list those values for which δ and n are non-negative integers and q is obtained from equation (3-10).

We only need to consider possibilities in which the factors have the same parity because

$(\delta - n + 89) + (\delta + n - 89) = 2\delta$

(Table-2)

When t_o =2

δ-(n-89)	δ +(n-89)	δ	n	q
-2	2440	1219	1310	663, 1272.5
-4	1220	608	701	663, 359
-8	610	301	398	359, 208.5
-20	244	112	221	123, 179
-40	122	41	170	123, 102.5
122	-40	41	8	-19, 1.5
-10	488	239	338	179, 298.5

So by lemma(1-3), there does not exist a (k,n;f) –arc of type ((n-19),n).

Hint: By similar proof, we checked the existence of (k,n;f) –arc of type (n-19,n) for all values from L_o=3 to L_o=170 and we found that (k,n;f)-arc does not exist for all these cases.

(4-2) The case when the number of points of weight zero is 171

In this case we discuss (k,n;f)-arc of type (n-19,n) in PG (2,9), when L_o =171.

The equation (3-10) becomes:

$2q^2 + (59-3n)q + n^2 - 22n + 285 = 0$ ……………….……..…....….(4-2-1)

From equation (3-11),we get

$(n-89)^2 - 2176 = \delta^2$

Hence

$(\delta - (n-89))(\delta + (n-89)) = -2176$

The possibilities of the factors of (–2176) are given below in table-3. We need to consider possibilities in which the factors have the same parity. Also we only list those value for which δ and n are non- negative integers, and q is obtained from equation (4-2-1).

(Table -3)

When t_o =171

δ-(n-89)	δ +(n-89)	δ	n	q
-2	1088	543	634	596.5, 325
-4	544	270	363	325, 190
-8	272	132	229	124, 190
-16	136	60	165	124, 94
136	-16	60	13	10, -20
-32	68	18	139	94, 85
68	-32	18	39	10, 19
64	-34	15	40	11.5, 19
-34	64	15	138	85, 92.5

Hence we can see from table-3 that the solutions of these compatible with n and δ being non negative integers and with q $\equiv$ 0 mod (n-m) are:

Case-1 when δ=15 and n=40, then from equation (3-7) & (3-8) L_1 =0 and L_2 =210, which is the ordinary case by definition (1-2).

Case -2 When δ=18 and n=39, lead to L_1 =20 and L_2 =190
i.e a (210,39;f) –arc of type (20,39) exist in PG(2,19). From lemma (1-3) and theorem (1-6) we get:
$400 \le W \le 419$ and $0 \le w \le 19$
When W=400 by theorem(1-4) we can get the following results:
$V_{20}^{0} = 20$ $\quad V_{39}^{0} = 0$
$V_{20}^{1} = 19$ $\quad V_{39}^{1} = 1$ ……….(4-2-2) $V_{20}^{2} = 18$
$V_{39}^{2} = 2$

From these results we have

(4-2-1) Theorem: the number of collinear points of weight zero are less than or equal to ten.
Proof: suppose that there is a 20-secant l such that l have 11 points of weight 0. Then the other 9 points on l has at most weight 2. Hence the weight of l is 18, which is a contraduction.

Corollary: the points of weight 0 form (171,10)-arc.

(4-3) classification of the lines of the plane with respect to an (210,39;f) –arc of type (20,39)

Let X be an 20-secant of (210,39;f)-arc of K. Since V_{39}^{0} = 0, then there only points of weight 2 and points of weight 1.Suppose that on X there are α points of weight 2 and β points of weight 1, then $\alpha + \beta = 20$ counting the points of weight 2 and 1 on X, we get $2\alpha + \beta = 39$.Hence the only solution of these equations is $\alpha = 19$ and $\beta = 1$.

Suppose that t_i, i =1,…,10 be a 20-secant having i points of weight 0 and α points of weight 2 and β points of weight 1

If i=1, then

$\alpha + \beta = 19$

$2\alpha + \beta = 20$

Hence the only solution of these equations is $\alpha = 1$, $\beta = 18$.

By the same way we found the solutions for i =2,…,10, see table -4

(Table-4)

Type of the line	Points of weight **0**	Points of weight **1**	Points of weight **2**
t_1	1	1	18
t_2	2	2	16
t_3	3	3	14
t_4	4	4	12
t_5	5	5	10
t_6	6	6	8
t_7	7	7	6
t_8	8	8	4
t_9	9	9	2
t_{10}	10	10	0
X	0	19	1

(4-3-1) Lemma

The lines of PG(2,19) are partitioned into eleven classes w.r.t. a minimal (210,39;f) –arc of type (20,39).

Proof: see table -4

Corollary-1: In all types from t_1 to t_{10}, we see that the number of the points of weight zero are the same as the number of the points of weight 2

Corollary-2: There is no point of weight 1 on the 10-secant of (171,10)-arc in PG(2,19).

Notation: By the same way we can check all the values of the points of weight zero for L_0=172,…,381, and since the method for finding any state of each cases is very complicated and need large number of procedure and after we check more than 170 cases, so it is very difficult to complete all the other cases for this reason we complete the research from L_0 = 340 to the last case.

(4-4) The case when L_0 = 340

In this case equation(3-10) becomes

$2q^2 + (59–3n)q + n^2 –22n – 621 = 0$

And from equation (3-11), we have

$\delta^2 = (n–89)^2 + 528$

Hence $(\delta – (n–89))(\delta +(n–89))=528$

The possibilities of the factors of (528) are given in table-5

(Table-5)

When t_o =340

δ-(n-89)	δ +(n-89)	δ	n	q
2	264	133	220	117, 183.5
4	132	68	153	117, 83
132	4	68	25	21, 13
8	66	37	118	83, 64.5
66	8	37	60	21, 39.5
22	24	23	90	47, 58.5
24	22	23	88	57, 45.5
44	12	28	73	47, 33
12	44	28	105	71, 57
88	6	47	48	33, 9.5
6	88	47	130	71, 94.5

So from (table-5) and by (lemma(1-5)), there does not exist a (k,n;f)-arc of type (n-19,n) where L_0 = 340.

We get the same result when L_0 = 341 see (table-6).

(Table-6)

When t_o =341

δ-(n-89)	δ +(n-89)	δ	n	q
2	272	137	224	119, 187.5
4	136	70	155	119, 84
136	4	70	23	20, -15
8	68	38	119	84, 65
68	8	38	59	20, 39
16	34	25	98	65, 52.5
34	16	25	80	39, 51.5

(4-5) The case when L_0 = 342

For this value of L_0 and form equation (3-10) we get

$2q^2 + (59–3n)q + n^2 – 22n – 625 = 0$ ……..(4-5-1)

and from (3-11) we have

$\delta^2 = (n–89)^2 + 560$

i.e $(\delta – (n –89))(\delta +(n–89))=560$

The possibilities of the factors of 560 are given in(table -7)

(Table -7)

When t_o =342

δ-(n-89)	δ +(n-89)	δ	n	q
2	280	141	228	121, 191.5
4	140	72	157	121, 85
140	4	72	21	19, -17
8	70	39	120	85, 65.5
70	8	39	58	19, 38.5
40	14	27	76	49, 35.5
14	40	27	102	55, 68.5
28	20	24	85	55, 43
20	28	24	93	49, 61
56	10	33	66	43, 26.5
10	56	33	112	61, 77.5

Further we need those value for which n and δ are non-negative integers.
Note that q is obtained from equation (4-5-1)
From (table -7) we have
δ = 72 and n =21 with q $\equiv$ 0 mod (n-m) so a (k,n;f) –arc of type (n-19,n) When L_0 = 342 exist
i.e a(39,21;f) –arc of type (2,21) and the points of weight 0 form (342,19)-arc
By similar method we can prove that a (k,n;f)-arc, for L_0 = 343,…,381 does not exist except when L_0 = 361.

(4-6) The case when L_0 = 361

Applying equation (3-10) with L_0 = 361 gives:

$2q^2 + (59–3n)q + n^2 – 22n – 663 = 0$

And from equation (3-11) we have

$(\delta – (n –89))(\delta +(n–89))=864$

The possibilities of the factors of 864 are given in (table-8)

(Table-8)

When t_o =361

δ-(n-89)	δ +(n-89)	δ	n	q
2	432	217	304	159, 267.5
4	216	110	195	159, 104
8	108	58	139	104, 75
108	8	58	39	0, 29
16	54	35	108	75, 57.5
54	16	35	70	29, 46.5
24	36	30	95	64, 49
36	24	30	83	40, 55
48	18	33	74	49, 32.5
18	48	33	104	55, 71.5
72	12	42	59	19, 40
12	72	42	119	64, 85
144	6	75	20	19, -18.5
6	144	75	158	85, 122.5

So a (k,n;f)-arc of type (n-19,n) when L_0 = 361 exist when δ =75 & n=20 that is (20,20;f) –arc of type (1,20) exist and the points of weight zero form (361,19)-arc.

(4-6-1) Theorem

A(k,n;f) –arc of type (n-19,n) in PG (2,19) does not exist for L_0 greater than 361.

Proof:

A (k,n;f)-arcs of type (n-19,n) exist if the points of the plane have at least three kinds of weights, i.e {0,1,w}, and we know that all the lines of the plane have a weight (n-19) except the 0-secants which have a weight n, that is, 0-secant does not contain any point of weight zero, means that a (k,n;f)–arc of type (n-19,n) have at least one 0-secant line and from the definition of the projective plane PG (2,19) we know that each line in the plane contains 20 points which means that the remaining points of G(2.19) is 381-20 = 361.

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