

## **A NEW HYBRID ALGORITHM FOR THE MODIFIED BARRIER FUNCTION FORM**

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Barrier

### **Abstract**

In this paper, we have developed a new formula which is more suitable to the Barrier function and we have also investigated a new hybrid algorithm of SD and VM formulas with exact line search in the field of non-linear constrained optimization.

## 1. INTRODUCTION:

### 1.1 A STANDARD BARRIER FUNCTION:

This approach is suitable for inequality constraints only.

the function  $\varphi(x, \rho)$  are of the form  $\varphi(x, \rho) = f(x) + \rho_k \sum_{i=1}^m \varphi\{c_i(x)\}$

denote  $\varphi_i = \varphi\{c_i(x)\}$

where

$\rho_k$  : Is appositve scalar.

$f(x)$  : Is the objective function .

$m$  : number of constraints.

$\varphi$  : Is the Barrier function .

There are many types of Barrier methods , see [ 1 , 4 , 3 ]  
such as :

$$B(x) = \sum_{j=1}^m \frac{1}{c_j(x)} \quad , \quad \dots\dots\dots(1)$$

$$B(x) = \sum_{j=1}^m \frac{1}{c_j(x)} \varepsilon \quad , \varepsilon > 0 \quad \dots\dots\dots(2)$$

$$B(x) = -\sum_{j=1}^m \ln(c_j(x)) \quad , \quad \dots\dots\dots(3)$$

Carroll in 1961 developed the inverse barrier function and in fact this idea further analytic and implemented by Fiacco and Mc. Cormic in 1964 and its known as the SUMT – method where SUMT stand for (sequential unconstrained minimization technique) the whole problem is now :

$$\min \varphi(x, \rho) = f(x) + \rho_k \sum_{i=1}^m 1/c_i(x) \quad \dots\dots\dots(4)$$

### 1.2 A STANDARD PENALTY FUNCTION:

Barrier function is not suitable for equality constrained because there is no way of ensuring that  $x$  satisfies all the constraints as equalities. Penalty function method follow similar procedure but the modification to the objective function comes into play when the constraints are violated and increases as the violation increases.

The Penalty function is of the form

$$\varphi(x, \rho) = f(x) + \frac{1}{\rho} \sum_{i=1}^m \varphi\{c_i(x)\} \quad \dots\dots\dots (5)$$

if  $\varphi_i = \varphi\{c_i(x)\}$

Then either  $\varphi_i = c_i^2(x)$  .....(6a)

this is usually used for the equality constraints or

$\varphi_i = [\min\{0, c_i(x)\}]^2$  ..... (6b)

this is used for the inequality constraints.

The function  $P(x): R^n \rightarrow R$  is called Penalty function to the equation

$$\begin{aligned} \min & f(x) \\ \text{s.t. } & g_j(x) \leq 0 \quad j = 1, \dots, m1 \\ & h_k(x) = 0 \quad k = 1, \dots, m2 \\ & x_i^l \leq x_i \leq x_i^u \quad i = 1, 2, \dots, n \quad , l, u : \text{upper and lower bounds} \end{aligned} \quad , \dots \dots \dots (7)$$

which is satisfies [8] :

$P(x) = 0$  if  $h(x) = 0$  ,  $c(x) \geq 0$  .....(8a)

$P(x) > 0$  if  $h(x) \neq 0$  or  $c(x) < 0$  .....(8b)

The general outlines of any algorithm used to solve such problems are as follows [7]:

1. Minimize objective as unconstrained function.
2. Provide penalty to limit constraint violations.
3. Magnitude of penalty varies through the optimization.
4. Called sequential unconstrained minimization technique (**SUMT**).
5. **create pseudo – objective**  $\phi(x, r_p) = f(x) + r_p P(x)$  Such that:

$f(x)$  = Original objective function.

$P(x)$  = Imposed Penalty function.

$r_p$  = Scalar multiplier to determine penalty magnitude .

$p$  = Unconstrained minimization number.

and there is four basic algorithms to solve such problem :

1. Exterior penalty function method.
2. Interior penalty function method.
3. Log penalty function method.
4. Extended interior penalty function method.

The effect of Penalty function is to create a local minimum of unconstrained problem near  $x^*$  .

Penalty function adds a penalty for infeasibility, barrier function adds a term that prevents iterates from becoming infeasible.

### 1.3 A MODIFIED APPROACH TO THE BARRIER AND PENALTY FUNCTIONS [9]:

Since the Barrier methods is a strictly methods,

$$P(x, r_p) = f(x) + r B(x) \dots\dots\dots (9)$$

Where  $f$  is a maximizing function and  $B$  is a barrier function. Consider the following:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq b_i \\ & x_i \geq 0 \end{aligned}$$

$$B(x) = \sum_{i=1}^m \frac{1}{b_i - g_i(x)} + \sum_{j=1}^n \frac{1}{x_j} \dots\dots\dots (10)$$

Remarks:

1. if  $b_i \longrightarrow g_i(x)$  then  $B(x) \longrightarrow \infty$ .
2. SUMT methods applied to the non-linear problems.
3. One of the most difficult tasks when using SUMT is the choice of the initial penalty parameters.

This is because the initial choice of penalty parameters is often a problem dependent and almost always has a major effect on overall optimization efficiency.

Therefore, we try to pick starting values for penalty parameters which will ensure a well-posed, efficient optimization task.

### 1.4 THE (SUMT) ALGORITHM:

The idea of SUMT methods is to solve the unconstrained  $\min p(x, r)$  Over a sequences of times. Each sequence reduces  $r$  i.e allows us to move closer to boundary. Usually go until some error tolerance is reached.

If

$$x_1 = \max(P(x, r), x^* \max f(x)) \dots\dots\dots (11)$$

then

$$f(x_1) \leq f(x^*) \leq f(x_1) + r.P(x_1) \dots\dots\dots (12)$$

make  $r_{new} = \rho.r$  , and let  $\rho = 0.1$  When the stopping criteria is

$$r \leq \varepsilon \dots\dots\dots (13)$$

### 1.5 OUTLINES OF SUMT ALGORITHM :

1. let  $x$  = initial solution (not on boundary) ,  $k=1$  ,  $\theta = 0.01$  ,  $r = 1$ .
2. at  $k$ -th iteraton ,

$x_{k-1}$  = Starting point, we can use multivariate unconstrained technique Such that gradient search method to find the local maximum / minimum of

$$p(x,r) = f(x) - r \cdot \left[ \sum_{i=1}^m \frac{1}{b_i - g_i(x)} + \sum_{j=1}^n \frac{1}{x_j} \right] \dots\dots\dots (14)$$

3. Stop if  $|x_k - x_{k-1}| \leq \varepsilon$
4. Else  $k = k+1$  and  $r = r.\theta$  go to step 2 .

## 2. EXTENDED INTERIOR PENALTY FUNCTION:

Incorporate the best features of the exterior and interior methods so that a penalty function is continuous every where and provides a sequence of improving feasible suboptimum. With  $g_0$  such that [5].

$$g_0 = -c(r_p)^a \quad \text{where} \quad \frac{1}{3} \leq a \leq \frac{1}{2}, \quad c = 0.2.$$

### 2.1 LINEAR EXTENDED PENALTY FUNCTION:

The penalty function is used for inequality constraints as follows:

$$P(x) = \sum_{j=1}^m g_j(x) \dots\dots\dots(15)$$

$$\text{if } g_j(x) \leq g_0 \text{ then } p_j(x) = \frac{-1}{g_j(x)} \dots\dots\dots(16)$$

$$\text{if } g_j(x) > g_0 \text{ then } p_j(x) = -\frac{2g_0 - g_j(x)}{g_0^2}, \dots\dots\dots(17)$$

and here we can define  $g_0$  as :

$$g_0 = -c(r_p)^a \dots\dots\dots(18)$$

Which is a small negative number, C is a constant determined at the beginning as follows:

Choose  $g_0$  in the range  $-0.3 \leq g_0 \leq -0.1$  then chose  $r_p$  such that:

$$r_p = |z| / \sum_{j=1}^m P_j(x) \dots\dots\dots(19)$$

then

$$c = -g_0 / (r_p)^a \dots\dots\dots(20)$$

## 2.2 QUADRATIC PENALTY FUNCTION:

Create a quadratic extended penalty function which has continuous second derivatives at  $g_j(x) = g_0$  and

$$p_j(x) = \frac{-1}{g_j(x)}, \quad g_j(x) \leq g_0$$

$$p_j(x) = -\frac{1}{g_0} \left\{ \left[ \frac{g_j(x)}{g_0} \right]^2 - 3 \left[ \frac{g_j(x)}{g_0} \right] + 3 \right\}, g_j(x) > g_0 \dots\dots\dots (21)$$

The quadratic extended penalty function may be more useful if a second-order method is used for unconstrained sub problems.

However, the degree of nonlinearity of  $\Phi$  is increased, when incorporating the extended penalty function into an optimization program we should provide several options and experiment to see which is more efficient for the class of problems we are solving.

## 3. A NEW GENERALIZED FORM TO THE DIFFERENCE BETWEEN THE CONSTRAINED

By taking the differences between the constrained in the tow cases when  $g_j(x) \leq g_0$  and  $g_j(x) > g_0$  and the extended penalty is linear we note that :

$$\begin{aligned}
 & \frac{-2g_0 + g_j}{g_0^2} + \frac{1}{g_j} \\
 &= \frac{g_j^2 - 2g_0g_j + g_0^2}{g_0^2g_j} \\
 &= \frac{-2g_0g_j + g_j^2 + g_0^2}{g_0^2g_j} \\
 &= \frac{(g_j - g_0)^2}{g_0^2g_j}, \dots\dots\dots(22)
 \end{aligned}$$

and in the case when extended penalty is quadratic:

The formula of the constrained when  $g_j(x) > g_0$  is the equation (21), and when we take the difference between the constraint in the two cases as the previous method, we note :

$$\begin{aligned}
 & -\frac{1}{g_0} \left\{ \left[ \frac{g_j}{g_0} \right]^2 - 3 \left[ \frac{g_j}{g_0} \right] + 3 \right\} + \frac{1}{g_j} \\
 &= -\frac{g_j^2}{g_0^3} + 3\frac{g_j}{g_0} - \frac{3}{g_0} + \frac{1}{g_j} \\
 &= \frac{-g_j^3 + 3g_j^2g_0 - 3g_jg_0^2 + g_0^3}{g_0^3g_j} \\
 &= -\frac{(g_j - g_0)^3}{g_0^3g_j} \dots\dots\dots(23)
 \end{aligned}$$

We note from the two previous cases the difference between the constraint when  $n = 1$  is as the equation (22) and when  $n = 2$  is as the equation (23), and from this we can conclude that the difference when  $n = 3$  to be as the form :

$$\frac{(g_j - g_0)^4}{g_0^4g_j} \dots\dots\dots(24)$$

By substituting this relation in the followed method (by taking the difference between the formula) we can conclude the formulae when  $g_j(x) > g_0$  and  $n = 3$  to be in the following form:

$$\frac{(g_j - g_0)^4}{g_0^4g_j} \text{cubic} + \frac{1}{g_j} =$$

$$\begin{aligned}
 \text{cubic} &= \frac{g_j^4 - 4g_j^3g_0 + 6g_j^2g_0^2 - 4g_jg_0^3 + g_0^4}{g_0^4g_j} - \frac{1}{g_j} \\
 &= \frac{g_j^4 - 4g_j^3g_0 + 6g_j^2g_0^2 - 4g_jg_0^3 + g_0^4 - g_0^4}{g_0^4g_j} \\
 &= \frac{g_j^3}{g_0^4} - 4\frac{g_j^2}{g_0^3} + 6\frac{g_j}{g_0^2} - 4\frac{1}{g_0} \\
 &= \frac{1}{g_0} \left\{ \left[ \frac{g_j(x)}{g_0} \right]^3 - 4 \left[ \frac{g_j(x)}{g_0} \right]^2 + 6 \left[ \frac{g_j}{g_0} \right] - 4 \right\}, \dots\dots\dots (25)
 \end{aligned}$$

and when  $n = 4$  and in the similar way, the general form to the difference is equal to :

$$\frac{(g_j - g_0)^5}{g_0^5g_j} \dots\dots\dots (26)$$

then the general form to the constrained when  $g_j(x) > g_0$  according the mathematical induction is as follows :

1. when  $n$  is odd:

$$\frac{(g_j - g_0)^{n+1}}{g_0^{n+1}g_j}, \dots\dots\dots (27)$$

2. when  $n$  is even:

$$\frac{(g_j - g_0)^{n+1}}{g_0^{n+1}g_j}, \dots\dots\dots (28)$$

### 3.1 A NEW GENERALIZED FORM OF THE CONSTRAINED FUNCTION WHEN $g_j(x) > g_0$

From our notation to the equations (22) ,(23), (24) , (26) we can conclude that the general form to the constrained when  $g_j(x) > g_0$  to be in the following way:



#### **4. A NEW HYBIRD BARRIER ALGORITHM:**

By using the new formula which is given in the equation (29), (30), and combine it with the standard algorithm we obtain the new hybrid algorithm and the outlines of this algorithm are as follow:

Step 1: take  $x_1$  as initial point ,  $k = 1$  .

Step 2: find  $z = f(x_k)$  and then find  $r_p = |z| / \sum_{j=1}^m p_j(x)$ .

Step 3: compute the sum of the constrained ( z1 , z2 ) as follows :  
 $\forall j = 1, \dots, m$  , s.t m : sum of constrained If  $g_j(x) > g_0$  compute z1  
as one of the form (29) , (30) and if  $g_j(x) \leq g_0$  compute z2 as  
equation (16), then find the sum z3 of the two cases.

Step 4: compute c as the equation (20).

Step 5: compute  $g_0 = -c(r_p)^a$ .

Step 6: H=I.

Step 7: find the value of function at the point  $x_1$  , and put it in  $f_1$ .

Step 8:  $d = -Hg_k$ .

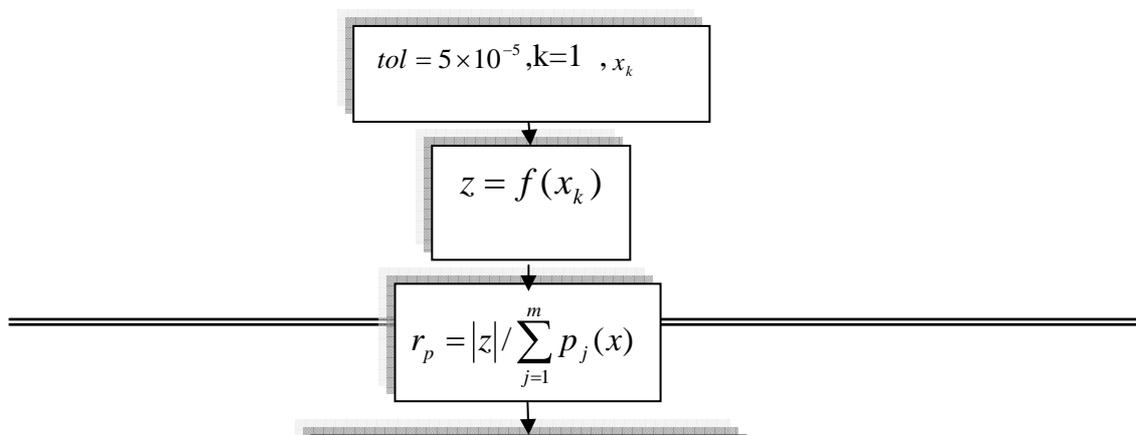
Step 9: use the cubic interpolation to find the step  $\lambda$  to compute  
 $x_{k+1} = x_k + \lambda.d_k$  .

Step 10: use the BFGS formulae to update H .

Step 11: comparison while  $|(f_2 - f_1) / f_1| < tol$  if this condition satisfy  
go to step 12 else put  $f_2$  in place of  $f_1$  and go to step 7.

Step 12: compares if  $r < tol$  stop else  $r = r/10$  and go to step 6 .

#### 4.1 FLOW CHART OF THE NEW HYBIRD ALGORITHM:



#### 4.2 Convergence of the new proposed algorithm;

Consider only the barrier methods (penalty methods can be analyzed in a similar way) applied to the problem

$$\begin{aligned} & \min f(x) \\ & \text{s.t. } g_i(x) \geq 0, i = 1, \dots, m \end{aligned} \quad \dots \quad (31)$$

Let  $S$  and  $S_0$  denote, respectively, the feasible region and its interior, i.e.

$$S = \{x \mid g_i(x) \geq 0, i=1, \dots, m\}, \dots\dots\dots (32)$$

$$S_0 = \{x \mid g_i(x) > 0, i = 1, \dots, m\} \dots\dots\dots (33)$$

### 4.3 Assumptions:

- (1)  $f$  and all  $g_i$  ( $i=1, \dots, m$ ) are continuous.  $\phi$  is a continuous function on  $S_0$ , and  $f(x) \rightarrow +\infty$  when  $x$  approaches the boundary of  $S$ .
- (2) For any  $a$ , the set  $\{x \mid x \in S, f(x) \leq a\}$  is bounded.
- (3)  $S_0$  is not empty.
- (4) Any  $y \in S$  can be approached by a sequence  $\{x_k\}$  in  $S_0$  :  

$$x_k \longrightarrow y, x_k \in S^0$$

### 4.4 Theorem (Convergence of the Barrier Methods)

Let  $\lim_{k \rightarrow \infty} \mu_k = 0$  ;  $B(x, \mu) = f(x) + \mu\phi(x)$  ;  $\mu_1 \geq \mu_2 \geq \dots$  and  $x_k$  is a global minimizer of problem  $\min_{k \rightarrow \infty} B(x, \mu_k)$ , then, for  $k=1, 2, \dots$

- (a)  $f(x_{k+1}) \leq f(x_k)$
- (b)  $\phi(x_{k+1}) \geq \phi(x_k)$
- (c)  $\{x_k\}$  has a convergent subsequence.
- (d) if the subsequence  $\{x_k \mid k \in K\}$  converges to  $\hat{x}$ , then  $\hat{x}$  must be a global solution of the problem (31).

**Proof:**

(a) For each  $k$ ,  $x_k$  is the solution of

$$\min_{x \in S^0} B(x, \mu_k) , \dots\dots\dots (34)$$

So,

$$B(x_k, \mu_k) \leq B(x_{k+1}, \mu_k) \Leftrightarrow f(x_k) + \mu_k \phi(x_k) \leq f(x_{k+1}) + \mu_k \phi(x_{k+1}) , \dots\dots\dots (35)$$

$$B(x_{k+1}, \mu_{k+1}) \leq B(x_k, \mu_{k+1}) \Leftrightarrow f(x_{k+1}) + \mu_{k+1} \phi(x_{k+1}) \leq f(x_k) + \mu_{k+1} \phi(x_k) , \dots\dots (36)$$

$$\mu_{k+1} \times (35) + \mu_k \times (36) ,$$

$$\Rightarrow \mu_{k+1} f(x_k) + \mu_k f(x_{k+1}) \leq \mu_{k+1} f(x_{k+1}) + \mu_k f(x_k)$$

$$\Rightarrow (\mu_k - \mu_{k+1}) f(x_{k+1}) \leq (\mu_k - \mu_{k+1}) f(x_k)$$

$$\Rightarrow (\mu_k - \mu_{k+1})(f(x_k) - f(x_{k+1})) \geq 0$$

$$\Rightarrow f(x_k) \geq f(x_{k+1}) \text{ where } (\mu_k \geq \mu_{k+1}) , \dots\dots\dots (37)$$

(b)

(37)-(35)

$$\Rightarrow \mu_k \phi(x_k) \leq \mu_k \phi(x_{k+1})$$

$$\Rightarrow \phi(x_k) \leq \phi(x_{k+1})$$

(c) Let  $S' = \{x \in S \mid f(x) \leq f(x_1)\}$  Since  $f(x_{k+1}) \leq f(x_k) \leq \dots \leq f(x_1)$   
 $\Rightarrow x_k \in S'$

As

continuity of  $f \Rightarrow S'$  is a closed set; and

assumption (2)  $\Rightarrow S'$  is a bounded set,

$\Rightarrow S'$  is a compact set.

So,  $\{x_k\}$  must have a convergent subsequence.

(d) Let  $x_k \xrightarrow{k} \hat{x}$  we need to prove that  $\hat{x}$  must be an optimal solution of problem (31).

(i) Since  $x_k \in S^0$ ,  $g_i(x_k) > 0 \quad (\forall k, \forall i)$  Then, as  $K \xrightarrow{k} \infty$ ,  $g_i(\hat{x}) \geq 0$ ,  
 $\forall i = 1, \dots, m. \Rightarrow \hat{x} \in S$ , i.e.  $\hat{x}$  is a feasible solution of problem (31).

(ii) Let  $x^*$  be a global minimizer of problem (31). By assumption (4), and the continuity of  $f$ , for any  $\varepsilon > 0$ , there exists  $x_\varepsilon \in S$  such that  
 $f(x_\varepsilon) < f(x^*) + \varepsilon$ .

Due to (31),  $B(x_k, \mu_k) \leq B(x_\varepsilon, \mu_k)$  i.e

$$f(x_k) + \mu_k \phi(x_k) \leq f(x_k) + \mu_k \phi(x_\varepsilon) < f(x_\varepsilon) + \varepsilon + \mu_k \phi(x_\varepsilon), \dots \dots \dots (38)$$

If  $\hat{x} \in S^0$ , then  $\phi(\hat{x})$  is a finite number, and  $\phi(x_k) \xrightarrow{k} \phi(\hat{x})$

Let  $k \xrightarrow{k} \infty$  from (36),

$$f(\hat{x}) + 0 \cdot \phi(\hat{x}) \leq f(x_\varepsilon) + \varepsilon + 0 \cdot \phi(x_\varepsilon) \text{ i.e}$$

$$f(\hat{x}) \leq f(x_\varepsilon) + \varepsilon, \dots \dots \dots (39.1)$$

If  $\hat{x} \notin S^0$ , i.e.  $\hat{x}$  is on the boundary of  $S$ , then  $\phi(\hat{x}) = +\infty$  So, for large  $k \in K$ ,  $\phi(x_k) > 0$  we have from (38),

$$f(x_k) \leq f(x_k) + \mu_k \phi(x_\varepsilon) < f(x_\varepsilon) + \varepsilon + \mu_k \phi(x_\varepsilon), \text{ for large } k \in K.$$

Let  $k \xrightarrow{k} \infty$  from the above inequalities, we have:

$$f(\hat{x}) \leq f(x_\varepsilon) + \varepsilon, \dots \dots \dots (39.2)$$

From (39.1) and (39.2), for any  $\varepsilon > 0$ ,  $f(\hat{x}) \leq f(x_\varepsilon) + \varepsilon$

$$\Rightarrow f(\hat{x}) \leq f(x_*)$$

$$\Rightarrow f(\hat{x}) = f(x_*)$$

$\Rightarrow \hat{x}$  is a global minimizer of problem (31).

## 5. NUMARICAL RESULTS:

Test of comparison contains (8) functions given in the appendix. All results obtained by using double precision by Pentium Computer and are programmed by Matlab language .

The efficiency of this algorithm depend on the number of function evaluation and number of iteration ,and results are explained in tables (5.1) , (5.2) .

The line search which is used in this algorithm is the cubic interpolation and the objective function and constraint are given for each test problem.

The stopping criteria which is used in all algorithms for all cases is  $r < 5 \times 10^{-5}$ .

Our numerical results shows that the new hybrid algorithm is effected when the comparison depend on NOI and NOF.

Table (5.1): Comparison between standard and hybrid algorithm.

Test function	Standard		Hybrid	
	NOI	NOF	NOI	NOF
1	11	43	20	107
2	Fail to converge		4	19

3	11	50	15	78
4	Fail to converge		16	80
5	Fail to converge		17	102
6	9	57	12	77
7	17	70	14	73
8	13	55	11	75

(5.1)

Table (5.2): if the standard algorithm take 100% of NOI,NOF

Tools	NOI	NOF
standard	100%	% 100
Hybrid	% 62	% 37

(5.2)

## 6. Appendix

1.

$$\min f(x) = (x_1 - 3)^2 + (x_2 - 4)^2$$

$$s.t. \quad 2x_1^2 - x_2^2 + 34 \geq 0$$

$$2x_1 - 3x_2 + 18 \geq 0$$

2.

$$\begin{aligned} \min f(x) &= -x_1x_2x_3 \\ \text{s.t. } 2x_1^2 + x_2^2 + 3x_3^2 - 51 &\geq 0 \end{aligned}$$

3.

$$\begin{aligned} \min f(x) &= x_1^2 + x_2^2 \\ \text{s.t. } x_1 + 1 &\geq 0 \\ x_2 + 1 &\geq 0 \end{aligned}$$

4.

$$\begin{aligned} \min f(x) &= -x_1^2 - x_2^2 \\ \text{s.t. } x_1^2 + x_2^2 - 17x_1 - 5x_2 + 66 &\geq 0 \\ x_1^2 + x_2^2 - 10x_1 - 10x_2 + 41 &\geq 0 \\ x_1^2 + x_2^2 - 4x_1 - 14x_2 + 45 &\geq 0 \\ x_1 + 7 &\geq 0 \\ x_2 + 7 &\geq 0 \end{aligned}$$

5.

$$\begin{aligned} \min f(x) &= (x_1 - x_2)^2 + (x_1 - 2x_2)^2 \\ \text{s.t. } -x_1^2 - x_2 &\leq 0 \end{aligned}$$

6.

$$\begin{aligned} \min f(x) &= x_1^2 + 3x_2^2 + 1.5x_3^2 \\ \text{s.t. } 2x_1 + x_2 + x_3 - 20 &\leq 0 \\ x_1 + x_3 - 10 &\leq 0 \end{aligned}$$

7.

$$\begin{aligned} \min f(x) &= 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2 \\ \text{s.t. } x_1^2 + x_2^2 + 5 &\geq 0 \\ 3x_1 - x_2 + 6 &\geq 0 \end{aligned}$$

8.

$$\begin{aligned} \min f(x) &= x_1^2 + x_2^2 \\ \text{s.t. } x_1^2 + x_2^2 &\leq 9 \\ x_1 + x_2 &\leq 1 \end{aligned}$$

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