

## Successive approximation method (S.A.M.) for solving integral equation of the first kind with symmetric kernel

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### **Abstract:**

In this paper we used successive approximate method (S.A.M.) to solve Fredholm integral equation of the first kind (F.I.E.1<sup>st</sup>. K.) with symmetric kernel. And also suggested an algorithm for this method the computer programming is given for the algorithm. The method and algorithm are tested on several numerical examples. After comparing the results with exact solution see tables (1, 2), it occurred that the results are good. (in this work Matlab programming used).

**Key words:** Symmetric kernel, Geometric series, successive approximation method and matlab program.

## 1-Introduction.

The mechanics problem of calculating the time a particle to slide under gravity down a given smooth curve, from any point on the curve to its lower end, leads to an exercise in integration. The time  $f(\tau)$  say, for the particle to descend from the height  $t$  is given by an expression of the form

$$f(\tau) = \int_0^{\tau} \frac{y(t)dt}{(\tau-t)^{\frac{1}{2}}}, \quad (a \leq \tau \leq b) \quad (1)$$

Where  $y(t)$  embodies the shape of the given curve.

The converse problem, in which the time of descent from height  $\tau$  is given and the particular curve which produces this time has to be found is less straightforward, as it entails the determination of the function  $\phi$  from (1),  $f(\tau)$  now being assigned for  $a \leq \tau \leq b$ . From this point of view (1) is called integral equation, this description expressing the fact that the function to be determined appears under an integral sign. The equation (1) is one of many integral equations which result directly from a physical problem. The notation adopted in this section, and throughout by  $y$  or  $y(s)$ .

Every integral equation contains a function obtained from  $\phi$  by integration and of the form  $\int_a^b k(s,t)\phi(t)dt$ , where  $k$  is called the kernel and is assumed known. For example in the integral equation

$$y(s) = \int_0^1 |s-t|y(t)dt + f(s) \quad , \quad (0 \leq s \leq 1) \quad (2)$$

the kernel is given by  $k(s,t) = |s-t|$ , and the function  $f(s)$  called the free term, is also assumed known.[2,9,12].

In general the kernel and free term will be complex-valued functions of real variables. A condition such as  $(0 \leq s \leq 1)$  following an equation indicates that the equation holds for all values of  $s$  in the given interval. Thus for the integral equation given above, we seek a solution  $y(s)$  satisfying the equation for all  $s$  in  $[0,1]$ .

Fredholm equation is therefore distinguished by having fixed finite limits of integration. We denote these limits by  $a$  and  $b$  here but we shall usually take  $a=0$  and  $b=1$  later, noting that the interval  $[0,1]$  can be transformed to general finite interval  $[a,b]$  by a simple change of variable.

The Fredholm integral equation of the first kind is

$$f(s) = \lambda \int_a^b K(s,t)y(t)dt \quad , \quad (a \leq s \leq b) \quad (3)$$

and the Fredholm integral equation of the second kind is

$$y(s) = f(s) + \lambda \int_a^b K(s,t)y(t)dt \quad , \quad (a \leq s \leq b) \quad (4)$$

where  $\lambda$  is a numerical parameter, Generally complex, in practical applications  $\lambda$  is usually composed of physical quantities. To solve the equation (3) there are several methods, like Galerkin, collocation [1, 2, 4, 9, 12]...etc. In this paper we try to solve type of equation (3) by S.A.M. .

**Definition. (Symmetric kernel).** [9, 11]

A function  $k(s,t)$  is said to be Symmetric kernel if  $k(s,t) = k(t,s)$  .

**Theorem (1):-**

If  $k(s,t)$  is Symmetric kernel and does not vanish at all points of S (S is a set) where it is continuous, then all of iterated functions  $k_2(s,t), k_3(s,t), \dots$  are Symmetric and none of them is identically zero.

Proof: - In [11].

## 2. Iterative kernels:-

The Fredholm Integral Equation has iterated kernels of the form.

Let  $K_1(s, t) = K(s, t) = k(t, s)$  which is symmetric kernel

$$K_2(s, t) = \int_a^b K(s, v) K_1(v, t) dv$$

$$K_3(s, t) = \int_a^b K(x, v) K_2(v, t) dv$$

$\vdots$

and in general

$$K_{n+1}(s, t) = \int_a^b K(s, v) K_n(v, t) dv \quad (5)$$

where  $n = 1, 2, 3, \dots, p$ . The function  $K_{n+1}(s, t)$  determined form formula (5) are called iterative kernels for them. [9].

### 3. Successive approximation method (S.A.M) to solve Fredholm integral equation of the first kind:-

The method of successive approximations consists in the following we have an integral equation

$$f(s) = \lambda \int_a^b K(s, t) y(t) dt . \quad (6)$$

The integral equation (6) may be solved by the method of successive approximations. To do this, put

$$y_n(s) = \sum_{n=0}^{\infty} \lambda^n u_n(s) \quad (7)$$

We construct a sequence of function  $\{y_n(s)\}$  with the aid of the recursion formula where the

$$u_1(s) = \int_a^b k(s, t) u_0(t) dt = \int_a^b K(s, t) f(t) dt. \quad (8)$$

$u_2(s)$  determined for the formula

$$\begin{aligned} u_2(s) &= \int_a^b K(s, t) u_1(t) dt \\ &= \int_a^b K(s, t) \left[ \int_a^b K(t, y) f(y) dy \right] dt \\ &= \int_a^b K_2(s, y) f(y) dy . \end{aligned} \quad (9)$$

Use (5) to find

$$K_2(s, y) = \int_a^b K(x, t) K_1(t, y) dt . \quad (10)$$

where  $K_1(t, y) = K(t, y) = k(y, t)$  because it is symmetric kernel. to find  $u_3(s)$

$$\begin{aligned} u_3(s) &= \int_a^b K(s, y) u_2(y) dy \\ &= \int_a^b K_3(s, y) f(y) dy. \end{aligned} \quad (11)$$

Use (5) to find

$$K_3(s, y) = \int_a^b K(x, t)K_2(t, y)dt \quad (12)$$

Similarly it is established that generally.

$$\begin{aligned} u_n(s) &= \int_a^b K(s, y)u_{n-1}(y)dy, n = 2,3,\dots, p \\ &= \int_a^b K_n(s, y)g(y)dy. \end{aligned} \quad (13)$$

Use (5) to find

$$K_n(s, y) = \int_a^b K(s, t)K_{n-1}(t, y)dt, n = 2,3,\dots, p \quad (14)$$

$K_n(s, y)$  Is called the  $n^{th}$  iterate kernel and  $K_n(s, y) = K_n(y, s)$  because by theorem (1) all it's iteration are symmetric.

In general

$$u_n(s) = \int_a^b K(s, t)u_{n-1}(t)dt = \int_a^b k_n(s, t)f(t)dt \quad (15)$$

where the function  $K_n(s, t)$  determined form formula (5)

The functions  $y_n(s)$  ( $n=1,2,3,\dots$ ) are considered as approximation to the derived solution of the equation; the zero approximation  $u_0(s)$  may be chosen arbitrary under conditions

$$\text{for } |\lambda| < \frac{1}{M}, \text{ where } M = \sqrt{\int_a^b \int_a^b k^2(s, t)dxdt} \quad (16)$$

The sequence (7) converges to the solution of equation (3).

The magnitude of the error of the  $(n+1)$ th approximation is given by the inequality

$$|y(s) - y_{n+1}(s)| \leq FC_1M^{-1} \frac{|\lambda M|^{n+1}}{1 - |\lambda M|} + \Psi C_1M^{-1} |\lambda M|^{n+1} \quad (17)$$

Where

$$F = \sqrt{\int_a^b y^2(s)ds}, \quad \Psi = \sqrt{\int_a^b y_0^2(s)ds}, \quad C_1 = \sqrt{\text{Max}_{a \leq s \leq b} \int_a^b k(s, t)dt}$$

**Algorithm of S.A.M.:-**

**Step 1:** Put  $u_0(s) = f(s)$  in the (F.I.E.1<sup>st</sup> .K.) with symmetric kernel.

**Step 2:** Find  $u_1(s)$  from equation (8).

**Step 3:** Put  $K_1(t, y) = K(t, y) = k(y, t)$  which is symmetric kernel

**Step 4:** Find  $K_n(t, y)$  from equation (5) for  $n = 2, 3, \dots, p$  which is symmetric kernel by theorem (1).

**Step 5:** Find  $u_n(s)$  from equation (11) for  $n = 2, 3, \dots, p$ .

**Step 6:** We compute  $y_n(s) = \sum_{n=0}^{\infty} \lambda^n u_n(s)$  which is approximate to the exact solution of (F.I.E.1<sup>st</sup> .K.) with symmetric kernel.

**4. Numerical examples and results:-**

The following examples are solved by using S.A.M.

**Example (1):-** Consider the integral equation  $y(s) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} k(s, t)u(t)dt$

where  $k(s, t) = \sin(s) \sin(t)$  which is symmetric kernel and  $y(s) = \frac{\sin(s)}{2}$

with the exact solution  $u(s) = \sin(s)$ .

**Sol.:-** Apply the S.A.M. in section (3) we obtained

$$u_0(s) = y(s) = \frac{\sin(s)}{2}$$

$$k_1(s, t) = \sin(s) \sin(t)$$

$$u_1(s) = \frac{\pi \sin(s)}{8}$$

$$k_2(s, t) = \frac{\pi \sin(s) \sin(t)}{4}$$

$$u_2(s) = \frac{\pi^2 \sin(s)}{32}$$

⋮

$$k_n(s, t) = \frac{\pi^{n-1} \sin(s) \sin(t)}{(4)^{n-1}}$$

$$u_n(s) = \frac{\pi^n \sin(s)}{2(4)^n}$$

Then  $y_n(s) = \sum_{n=0}^{\infty} \mathcal{L}^n u_n(s)$  which is geometric series and its convergence. The results for example (1) is shown in table (1).

**Table (1): the results of difference value of  $s$  and  $p$ .**

s	$p$	Exact solution	Value by S.A.M.	Error	Running time.
0.314159	1	0.309016	0.231760	0.077254	0.040000
	3	0.309016	0.289703	0.019313	0.172000
	5	0.309016	0.304188	0.004828	0.406000
	7	0.309016	0.307809	0.001207	0.656000
	9	0.309016	0.307815	0.000301	1.047000
	11	0.309016	0.308941	0.000075	1.547000
	13	0.309016	0.308998	0.000018	2.266000
	15	0.309016	0.309012	0.000004	3.125000
	17	0.309016	0.309015	0.000000	4.015000
19	0.309016	0.309016	0.000000	5.218000	
0.785398	1	0.707106	0.530330	0.176776	0.040000
	3	0.707106	0.662912	0.044194	0.172000
	5	0.707106	0.696058	0.011048	0.406000
	7	0.707106	0.704344	0.002742	0.656000
	9	0.707106	0.706416	0.000690	1.047000
	11	0.707106	0.706934	0.000172	1.547000
	13	0.707106	0.707063	0.000043	2.266000
	15	0.707106	0.707096	0.000010	3.125000
	17	0.707106	0.707104	0.000002	4.015000
19	0.707106	0.707105	0.000001	5.218000	
1.256637	1	0.951056	0.713292	0.237764	0.040000
	3	0.951056	0.891615	0.059441	0.172000
	5	0.951056	0.936196	0.014860	0.406000
	7	0.951056	0.947341	0.003715	0.656000
	9	0.951056	0.950127	0.000928	1.047000
	11	0.951056	0.950824	0.000232	1.547000
	13	0.951056	0.959984	0.000058	2.266000
	15	0.951056	0.951042	0.000014	3.125000
	17	0.951056	0.951052	0.000003	4.015000
19	0.951056	0.951054	0.000001	5.218000	

**Example (2):-** Consider the integral equation  $y(s) = \frac{5}{2} \int_0^1 k(s,t)u(t)dt$

where  $k(s,t) = s^2t^2$  which is symmetric kernel and  $y(s) = \frac{s^2}{4}$

with the exact solution  $u(s) = \frac{s^2}{2}$ .

Sol.:- Apply the S.A.M. in section (3) we obtained

$$u_0(s) = y(s) = \frac{s^2}{4}$$

$$k_1(s,t) = s^2t^2$$

$$u_1(s) = \frac{s^2}{20}$$

$$k_2(s,t) = \frac{s^2t^2}{5}$$

$$u_2(s) = \frac{s^2}{100}$$

⋮

$$k_n(s,t) = \frac{s^2t^2}{(5)^{n-1}}$$

$$u_n(s) = \frac{s^2}{2(5)^n}$$

Then  $y_n(s) = \sum_{n=0}^{\infty} \lambda^n u_n(s)$  which is geometric series and its convergence.

The results for example (2) is shown in table (2).

Table (2): the results of difference value of  $S$  and  $p$ .

$s$	$p$	Exact solution	Values by S.A.M.	Error	Running time.
0.2	1	0.020000	0.015000	0.005000	0.036000
	3	0.020000	0.017800	0.001250	0.156000
	5	0.020000	0.019687	0.000312	0.328000
	7	0.020000	0.019921	0.000078	0.578000
	9	0.020000	0.019980	0.000019	0.906000
	11	0.020000	0.019995	0.000004	1.312000
	13	0.020000	0.019998	0.000001	1.797000
	15	0.020000	0.0199996	0.000000	2.344000
	17	0.020000	0.0199999	0.000000	3.000000
	19	0.020000	0.0199999	0.000000	3.565000
0.5	1	0.125000	0.093700	0.031250	0.036000
	3	0.125000	0.117178	0.007812	0.156000
	5	0.125000	0.123046	0.001953	0.328000
	7	0.125000	0.124511	0.000488	0.578000
	9	0.125000	0.124877	0.000122	0.906000
	11	0.125000	0.124969	0.000030	1.312000
	13	0.125000	0.124992	0.000019	1.797000
	15	0.125000	0.124998	0.000004	2.344000
	17	0.125000	0.124998	0.000001	3.000000
	19	0.125000	0.124999	0.000000	3.565000
0.8	1	0.320000	0.240000	0.080000	0.036000
	3	0.320000	0.300000	0.020000	0.156000
	5	0.320000	0.315000	0.005000	0.328000
	7	0.320000	0.318750	0.001250	0.578000
	9	0.320000	0.319687	0.000312	0.906000
	11	0.320000	0.319921	0.000078	1.312000
	13	0.320000	0.319998	0.000019	1.797000
	15	0.320000	0.319995	0.000004	2.344000
	17	0.320000	0.319998	0.000001	3.000000
	19	0.320000	0.319999	0.000000	3.565000

## 5- Conclusion.

After testing the SM and its Algorithm on several Numerical examples the results are obtained are obvious in tables (1, 2), they are very good results, for instance (1) in table (1) for solving example (1) shows at  $s=0.314159$ , at  $n=17$  iteration we obtain approximation solution with error 0.000000 at time 4.0150 second, and also for after values of  $s$  results are clear. In the table (2) for solving example (2) shows at  $s=0.2$ , at  $n=13$  iteration error becomes 0.000001 and RT equal 1.797000 second, is also the excellent, results and it's clear for different values of  $s$ .

All of the numerical examples gave good results, but in this paper, we occurred only two examples. But results indicated the successive approximation method, was successively for solving (F.I.E.1<sup>st</sup>. K.) with symmetric kernel.

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