Journal of Education and Science (ISSN 1812-125X)

www.edusj.uomosul.edu.iq

Numerical Solution of Modified Cubic-Boussinesq Equation using He-Kamal Transform with Lagrange Multiplier

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Article information

Article history:
Received: September 20, 2024
Accepted: January 04, 2025
Available online: April 01, 2025

Keywords:
Cubic-Boussinesq
Lagrange multiplier
He-Kamal transform
Wave theory

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Abstract

Cubic-Boussinesq equation is very crucial for modeling nonlinear wave phenomena, capturing intricate dynamics like wave breaking and soliton interactions. Its significance lies in its ability to describe the behavior of waves in diverse physical contexts, from fluid dynamics to optical fibers. Due to the nonlinearity in the equation, finding accurate and efficient solutions might be quite challenging. This study introduces an innovative approach using the He-Kamal transform method and a Lagrange multiplier to solve the equation. The He-Kamal transform simplifies the PDE, making it more tractable, while the Lagrange multiplier enhances solution accuracy and convergence. Numerical simulations show that the He-Kamal transform with a Lagrange multiplier corresponds with traditional methods in handling the cubic nonlinearity of the Cubic-Boussinesq equation. MATLAB-generated diagrams demonstrate the effectiveness of the method in capturing wave dynamics and stability. This research advances numerical techniques for solving nonlinear PDEs and contributes to the field of nonlinear wave theory.

DOI: 10.33899/edusj.2025.153356.1500, ©Authors, 2025, College of Education for Pure Science, University of Mosul.

1. Introduction

Nonlinear wave equations are crucial for understanding a wide range of physical phenomena, from fluid dynamics to optical fiber communication [4]. The Boussinesq equation is an important formula for simulating long wave propagation on shallow water surfaces [9, 14]. By adding a cubic nonlinear element to the traditional Boussinesq formulation, the Cubic-Boussinesq equation expands to more accurately represent complex dynamics like wave breaking and soliton interactions [10, 12]:

$$w_{tt} - [D(w)]_{qq} - w_{qqqq} = f(q,t), \qquad (1.1)$$

Where $f(q,t)$ represents a source function and $D(w)$ represents an arbitrary sufficiently differentiable function, with the requirement that $[D(w)]_{qq} \neq 0$ to assure nonlinearity. Here are the initial conditions: $w(q,0) = g_1(q), w_t(q,0) = g_2(q)$.

The conventional methods like the Variational Iteration Method [3, 15], Adomian Decomposition Method [6, 16], and Homotopy Perturbation Method (HPM) [8, 19] combined with integral transforms [17, 18]. These methods have been widely used because of their adaptability and durability while solving nonlinear partial differential equations. In order to iteratively refine solutions, HPM uses a perturbation series [7, 8]. To approximate answers through repeated adjustments, VIM uses the variational principle. Differential equations are simplified for easier solutions using integral transform methods like the Laplace or Fourier transforms. While still useful, these traditional approaches can be computationally demanding and complex to handle significant nonlinearities.

In this work, we investigate a novel numerical method that uses the He-Kamal transform method in conjunction with a Lagrange multiplier to solve the Cubic-Boussinesq equation. The He-Kamal transform makes the original PDE easier to handle numerically by reducing it to a more manageable form. The accuracy and convergence of the solution are further improved by adding a Lagrange multiplier, especially when handling the cubic non- linearity of the equation.

2. Preliminaries

2.1 Kamal Transform Method

Definition 2.1. A new transform known as the Kamal transform will be examined for functions with an exponential order. It is defined by [1, 2]:

$$A = \left\{ Q(t) : \exists, M, c_1, c_2 > 0. |Q(t)| < M e^{\frac{|t|}{c_j}}, if\ t \in (-1)^j \times [0, \infty) \right\}, \tag{2.1}$$

$$\exists M \in \mathbb{R}^+ \forall a \in A, |c_1(a)| \leq M\ and\ c_2(a) \in \mathbb{R} \cup \{\infty\}$$

Kamal transform is represented as $\mathcal{K}$,

$$\mathcal{K}\{Q(t)\} = \mathcal{Q}(v) = \int_0^\infty Q(t) e^{-\frac{t}{v}} dt. t \geq 0,\ c_1 \leq v \leq c_2 \tag{2.2}$$

Remark 2.2. The Kernel of the Kamal transform is written as $e^{-\frac{t}{v}}$

2.2 Properties of Kamal Transform

Theorem 2.3 (Linearity Property [1, 2]. Let $Q_1(t)$ and $Q_2(t)$be two distinct functions, and Kamal Transform is $Q_1(v)$ and $Q_2(v)$ respectively, then

$$\mathcal{K}[c_1 Q_1(t) + c_2 Q_2(t)] = c_1 \mathcal{K} Q_1(t) + c_2 \mathcal{K} Q_2(t) = c_1 Q_1(v) + c_2 Q_2(v) \tag{2.3}$$

c_1 and c_2 are constants.

Proof. By definition, the Kamal Transform gives us

$$\mathcal{K}\{Q(t)\} = \int_0^\infty Q(t)\, e^{-\frac{t}{v}} dv \tag{2.4}$$

$$\mathcal{K}[c_1 Q_1(t) + c_2 Q_2(t)] = \int_0^\infty [c_1 Q_1(t) + c_2 Q_2(t)]\, e^{-\frac{t}{v}} dv$$

$$= c_1 \left[\int_0^\infty Q_1(t)\, e^{-\frac{t}{v}} dv \right] + c_2 \left[\int_0^\infty Q_2(t)\, e^{-\frac{t}{v}} dv \right]$$

∴ Integral is a linear operator,

$$= c_1\mathcal{K}[Q_1(t)] + c_2\mathcal{K}[Q_2(t)],$$
$$= c_1Q_1(v) + c_2Q_2(v) \qquad (2.5)$$

where c_1, c_2 are constants.

Theorem 2.4 (Integral Function [1,2]. *If* $\mathcal{K}[Q(t)] = Q(v)$, then $\mathcal{K}\left\{\int_0^1 F(t)dt\right\} = vQ(v)$

Proof: Let $H(t) = \int_0^1 F(t)dt$. Then $\acute{H}(t)$ and $H(0) = 0$. The Kamal Transform of the derivative of function characteristic gives:

$$\mathcal{K}\left[\acute{H}(t)\right] = \frac{1}{v}\mathcal{K}\{H(t)\} - H(0) = \frac{1}{v}\mathcal{K}\{H(t)\}$$
$$\mathcal{K}[H(t)] = v\mathcal{K}\left\{\acute{H}(t)\right\} = v\mathcal{K}\{F(t)\}$$

$$\mathcal{K}[H(t)] = vG(v)$$
$$\mathcal{K}\left\{\int_0^\infty F(t)dt\right\} = vG(v) \qquad (2.5)$$

Theorem 2.5 (Differential Property [1, 2]. If $\mathcal{K}[Q(t)] = Q(v)$, then

1. $\mathcal{K}\{Q'(t)\} = \frac{1}{v}Q(v) - Q(0)$
2. $\mathcal{K}Q''(t)\left(= \frac{1}{v^2}Q(v) - \frac{1}{v}Q(0) - Q'(0)\right.$
3. $\mathcal{K}\left\{Q^{(r)}(t)\right\} = \frac{1}{v^r}Q(v) - \sum_{k=0}^{r-1} v^{k-r+1}Q^k(0)$

Table 2.1: Kamal Transform of Some Frequently Encountered Functions [1, 2]

No	$Q(t)$	$K\{Q(t)\} = Q(v)$
1.	1	v
2.	t	v^2
3.	t^2	$2!\ v^3$
4.	$t^r, r \in R$	$r!\, v^{r+1}$
5.	$t^r, r > -1$	$\Gamma(r+1)v^{r+1}$
6.	e^{kt}	$\frac{v}{1-kv}$
7.	$sinkt$	$\frac{kv^2}{1+k^2v^2}$
8.	$coskt$	$\frac{v}{1+k^2v^2}$
9.	$sinhkt$	$\frac{kv^2}{1-k^2v^2}$
10.	$coshkt$	$\frac{v}{1-k^2v^2}$

Table 2.2: Inverse Kamal Transform

No	$Q(t)$	$K\{Q(t)\} = Q(v)$
1.	v	1
2.	v^2	t
3.	v^3	$\frac{t^2}{2!}$
4.	$v^{r+1}, r \in N$	$\frac{t^r}{r!}$
5.	$v^{r+1}, r > -1$	$\frac{t^r}{\Gamma(r+1)}$
6.	$\frac{v}{1-kv}$	e^{kt}
7.	$\frac{v^2}{1+k^2v^2}$	$\frac{sinkt}{k}$
8.	$\frac{v}{1+k^2v^2}$	$coskt$
9.	$\frac{v^2}{1-k^2v^2}$	$sinhkt$
10.	$\frac{v}{1-k^2v^2}$	$coshkt$

2.3 He Polynomial

The He polynomial is an iterative technique introduced by Ji-Huan He in 1999 for solving nonlinear differential equations. It combines the concept of homotopy from topology with the traditional perturbation method, enabling the solution of a wide range of linear and nonlinear problems without requiring small parameters, which are usually needed in classical perturbation methods [7, 8].

The method constructs a homotopy that continuously transforms a difficult-to-solve nonlinear problem into a simpler problem with a known solution. Let the nonlinear differential equation be:

$$A(w) - f(q) = 0,$$

where $A(w)$ is a nonlinear operator, $f(x)$ is a known function, and uu is the unknown function. A homotopy $H(v,p)$ is constructed as:

$$H(v,p) = (1-p)[L(v) - L(w_0)] + p[A(v) - f(q)],$$

where:

i. $p \in [0,1]$ is the homotopy parameter,
ii. L is a linear operator,
iii. u_0 is the initial approximation (an easily solvable case).

When $p = 0, H(v,0)$ corresponds to the linear problem $L(v) - L(w_0) = 0$, and when $p = 1, H(v,1)$ corresponds to the original nonlinear problem

$$A(v) - f(q) = 0$$

The solution v is expressed as a power series in p:

$$v = w_0 + pw_1 + p^2 w_2 + \cdots$$

Substituting this expansion into the homotopy $H(v,p) = 0$, the terms are collected by powers of p, yielding a sequence of linear equations for $w_0, w_1, \dots, w_n$

In HPM, He polynomials are used to systematically manage and simplify the higher-order nonlinear terms in the series. The He polynomials for a nonlinear function $N(u)$ are defined as:

$$H_n(w_0, w_1, \dots, w_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n} N \sum_k^{\infty} p^k w_k$$

When $p = 0$

These polynomials allow for the expansion of $N(u)$ as:

$$N \sum_{k=0}^{\infty} w_k = \sum_{n=0}^{\infty} H_n(w_0, w_1, \dots, w_n)$$

$$H_0(w) = w_0^3$$

$$H_1(w) = w_0^2 w_1$$

$$H_2(w) = 3w_0 w_0^2 3w_0^2 w_2 \tag{2.6}$$

Using the recursive relation derived from collecting terms with the same powers of p, the approximate solution is obtained by summing up the terms:

$$w = w_0 + w_1 + w_2 + \cdots$$

2.4 Lagrange Multiplier

The Lagrange multiplier is a crucial component of the Variational Iteration Method (VIM), which Ji-Huan He introduced to provide analytical approximations to linear and nonlinear problems. This multiplier facilitates the construction of correction functionals that iteratively converge to the solution [20].

The Lagrange multiplier is a function or constant determined based on variational theory. In the context of VIM, it is used to optimize the correction function to satisfy the system's governing equations. It ensures that the constraints imposed by the differential equation are respected throughout the iterative process.

Consider a differential equation of the form [20]:

$$L(w) + N(w) = g(q),$$

where:

- L is a linear operator,
- N is a nonlinear operator,
- $g(q)$ is a source term,
- $w(q)$ is the unknown function.

The correction functional in VIM is constructed as:

$$w_{n+1}(q) = w_n(q) + \int_0^x \lambda(v)\left[L\big(w_n(v)\big) + N\big(\widehat{w}_n(v)\big) - g(v)\right] dv,$$

where:

- $w_n(q)$ is the approximation in the nth iteration,
- $\lambda(v)$ is the Lagrange multiplier to be determined,
- $\widehat{w}_n(v)$ is a restricted variation, satisfying $\delta\widehat{w}_n(v)$.

The Lagrange multiplier $\lambda(v)$ is obtained using variational theory. By ensuring that the correction functional satisfies the Euler-Lagrange equations of the problem, $\lambda(v)$ is derived explicitly. For most linear problems, the multiplier $\lambda(v)$ is a constant, while for nonlinear problems, it may depend on s and other parameters.

2.4 Construction of the Proposed Method

Given the following partial differential equation [13]

$$L(w) + N(w) - q(m) = 0 \quad (2.7)$$

Where

$L(w)$ is a Linear term

$N(w)$ is a nonlinear term

$Q(m)$ is the source term

$W(q,t)$ is the unknown function

Taking the Kamal transform of equation (2.6)

$$\mathcal{K}[L(w) + N(w) - q(m)] = 0 \quad (2.8)$$

Now let's apply the Lagrange multiplier to the above equation (2.7), $\lambda\ (v)$, we get

$$\lambda(v)\mathcal{K}[L(w) + N(w) - q(m)] = 0 \quad (2.9)$$

$$\lambda(v)\{\mathcal{K}[L(w) + N(w) - q(m)]\} = 0 \quad (2.10)$$

Thus, the recurrence relation is transformed to:

$$w_{r+1}(q,v) = w_r(q,v) - \lambda\{\mathcal{K}[L(w) + N(w) - q(m)]\} \quad (2.11)$$

Using the Kamal transform to determine the value of the Lagrange multiplier $\lambda(v)$, we show that w_r is a restricted variable, i.e., $\delta w_r = 0$ and

$$\frac{\delta w_{r+1}(q,v)}{\delta w_r(q,v)} = 0 \quad (2.12)$$

Taking the inverse Kamal transform of (2.10), yields

$$w_{r+1}(q,t) = w_r(q,t) - \lambda\{\mathcal{K}[L(w) + N(w) - q(m)]\} \quad (2.13)$$

Lastly, the approximate solution is investigated using the He-Kamal with Lagrange multiplier by changing the computed coefficients from equation (2.12).

Note that, to simplify the nonlinearity$N(w_r)$, the He polynomial (2.6) is used

3. Main Result

Example 3.1. Consider the cubic-Boussinesq equation of the form [14, 12]

$$w_{tt} - w_{qq} + 2(w^3)_{qq} - w_{qqqq} = 0 \quad (3.1)$$

With

$$w(q,0) = \frac{1}{q}, \quad w_t(q,0) = -\frac{1}{q^2} \quad (3.2)$$

Applying the Kamal Transform to (3.1), then, the equation becomes

$$\mathcal{K}\left[\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial q^2} + 2\frac{\partial^2}{\partial q^2}(w^3) - \frac{\partial^4 w}{\partial q^4}\right] = 0, \quad (3.3)$$

using the recurrence relation (2.12) to equation (3.3), then we get

$$w_{r+1}(q,v) = w_r(q,v) + \lambda(v)\mathcal{K}\left[\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial q^2} + 2\frac{\partial^2 w}{\partial q^2}(w^3) - \frac{\partial^4 w}{\partial q^4}\right], \quad (3.4)$$

$$\delta w_{r+1}(q,v) = w_r(q,v) + \lambda(v)\left[\left(\frac{1}{v^2}w(q,v) - \frac{1}{v}w(q,0) - w'(q,0)\right) - \mathcal{K}\left(\frac{\partial^2 w}{\partial q^2} - 2\frac{\partial^2}{\partial q^2}(w^3) + \frac{\partial^4 w}{\partial q^4}\right)\right], \quad (3.5)$$

this implies that

$$\delta w_{r+1}(q,v) = \delta w_r(q,v) + \frac{1}{v^2}\lambda \delta w_r(q,v), \quad (3.6)$$

Consequently,

$$\lambda(v) = -v^2 \quad (3.7)$$

Notice that w_r is a restricted variable, $\delta w_r = 0$

$$\frac{\delta w_{r+1}(q,v)}{\delta w_r(q,v)} = 0 \quad (3.8)$$

the value of $\lambda(v)$ is used in equation (3.7)

$$w_{r+1}(q,v) = w_{r+1}(q,v) - v^2\mathcal{K}\left(\frac{\partial^2 w}{\partial q^2} - 2\frac{\partial^2}{\partial q^2}(w^3) + \frac{\partial^4 w}{\partial q^4}\right) \quad (3.9)$$

Now, taking inverse Kamal of equation (3.9)

$$w_{r+1}(q,t) = w_r(q,t) - \mathcal{K}^{-1}\left[v^2\mathcal{K}\left(\frac{\partial^2 w}{\partial q^2} - 2\frac{\partial^2}{\partial q^2}(w^3) + \frac{\partial^4 w}{\partial q^4}\right)\right] \quad (3.10)$$

Applying He's polynomial (2.13) to equation (3.10), we get

$$\begin{aligned} w_0 + \rho w_1 + \rho^2 w_2 + \rho^3 w_3 + \cdots \ldots \\ = w_r(q,t) + \rho\mathcal{K}^{-1}\left[v^2\mathcal{K}\left(\frac{\partial^2 w_0}{\partial q^2} - 2\frac{\partial^2}{\partial q^2}(w_0^3) + \frac{\partial^4 w_0}{\partial q^4}\right)\right] \\ + \rho^2\mathcal{K}^{-1}\left[v^2\mathcal{K}\left(\frac{\partial^2 w_1}{\partial q^2} - 2\frac{\partial^2(w_0^2 w_1)}{\partial q^2} + \frac{\partial^4 w_1}{\partial q^4}\right)\right] \\ + \rho^3\mathcal{K}^{-1}\left[v^2\mathcal{K}\left(\frac{\partial^2 w_2}{\partial q^2} - 2\frac{\partial^2(3w_0 w_1^2 + 3w_0^2 w_2)}{\partial q^2} + \frac{\partial^4 w_2}{\partial q^4}\right)\right] \end{aligned}$$

Equating the coefficients of the like powers of ρ yields:

$\rho^0 \quad : \quad w_0 = w_0(q,t) + t w_0(q,t),$

$\rho^1 \quad : \quad w_1 = \mathcal{K}^{-1}\left[v^2\mathcal{K}\left(\frac{\partial^2 w_0}{\partial q^2} - 2\frac{\partial^2}{\partial q^2}(w_0^3) + \frac{\partial^4 w_0}{\partial q^4}\right)\right],$

$\rho^3 \quad : \quad w_2 = \mathcal{K}^{-1}\left[v^2\mathcal{K}\left(\frac{\partial^2 w_0}{\partial q^2} - 6\frac{\partial^2}{\partial q^2}(w_0^2 w_1) + \frac{\partial^4 w_0}{\partial q^4}\right)\right].$

Applying the initial and boundary value of the Problem

$$w_0(q,t) = \frac{1}{q} - \frac{t}{q^2}, \quad (3.11)$$

$$w_1(q,t) = \mathcal{K}^{-1}\left[v^2\mathcal{K}\left(\frac{\partial^2 w_0}{\partial q^2} - 2\frac{\partial^2}{\partial q^2}(w_0^3) + \frac{\partial^4 w_0}{\partial q^4}\right)\right],$$

$$= \mathcal{K}^{-1}\left[v^2\mathcal{K}\left(\frac{\partial^2}{\partial q^2}\left(\frac{1}{q}-\frac{t}{q^2}\right)-2\frac{\partial^2}{\partial q^2}\left(\frac{1}{q}-\frac{t}{q^2}\right)^3+\frac{\partial^4}{\partial q^4}\left(\frac{1}{q}-\frac{t}{q^2}\right)\right)\right],$$

$$= \mathcal{K}^{-1}\left[v^2\mathcal{K}\left(\frac{2}{q^3}-\frac{6t}{q^4}-2\left(\frac{12}{q^5}-\frac{60t}{q^6}+\frac{126t^2}{q^8}-\frac{42t^3}{q^8}\right)+\frac{30}{q^6}-\frac{144t}{q^7}\right)\right],$$

$$= \mathcal{K}^{-1}\left[v^2\mathcal{K}\left(\frac{2}{q^3}-\frac{6t}{q^4}-\frac{24}{q^5}+\frac{120t}{q^6}-\frac{252t^2}{q^8}+\frac{84t^3}{q^8}+\frac{30}{q^6}-\frac{144t}{q^7}\right)\right],$$

$$= \mathcal{K}^{-1}\left[v^2\left(\frac{2v}{q^3}-\frac{6v^2}{q^4}-\frac{24v}{q^5}+\frac{120v^2}{q^6}-\frac{504v^3}{q^8}+\frac{252v^4}{q^8}+\frac{30v}{q^6}-\frac{144v^2}{q^7}\right)\right],$$

$$= \mathcal{K}^{-1}\left[\left(\frac{2v^3}{q^3}-\frac{6v^4}{q^4}-\frac{24v^3}{q^5}+\frac{120v^4}{q^6}-\frac{504v^5}{q^8}+\frac{252v^6}{q^8}+\frac{30v^3}{q^6}-\frac{144v^4}{q^7}\right)\right],$$

$$= \frac{t^3}{q^3}-\frac{t^3}{q^4}-\frac{12t^2}{q^5}+\frac{20t^3}{q^6}-\frac{21t^4}{q^8}+\frac{21t^5}{10q^8}+\frac{5t^2}{q^6}-\frac{24t^3}{q^7}$$

where $-\frac{12t^2}{q^5}+\frac{20t^3}{q^6}-\frac{21t^4}{q^8}+\frac{21t^5}{10q^8}+\frac{5t^2}{q^6}-\frac{24t^3}{q^7}$ are noise terms,

$= \frac{t^3}{q^3}-\frac{t^3}{q^4}$ (3.12)

$w_2 = \mathcal{K}^{-1}\left[v^2\mathcal{K}\left(\frac{\partial^2 w_0}{\partial q^2}-6\frac{\partial^2}{\partial q^2}(w_0^2 w_1)+\frac{\partial^4 w_0}{\partial q^4}\right)\right],$

$= \frac{t^4}{q^5}-\frac{t^5}{q^6}$ (3.13)

Hence, the solution of equation (3.1) is expressed as

$$w_r(q,t) = w_0 + w_1 + w_2 + \ldots$$

$= \frac{1}{q}-\frac{t}{q^2}+\frac{t^3}{q^3}-\frac{t^3}{q^4}+\frac{t^4}{q^5}-\frac{t^5}{q^6}+\cdots\ldots$ (3.14)

$= \sum_{r=0}^{\infty}\frac{(-t)^r}{q^{r+1}}$ (3.15)

The closed form of equation (3.15) is

$w(q,t) = \frac{1}{q+t}$ (3.16)

The result is the same as the result obtained in [14, 12]

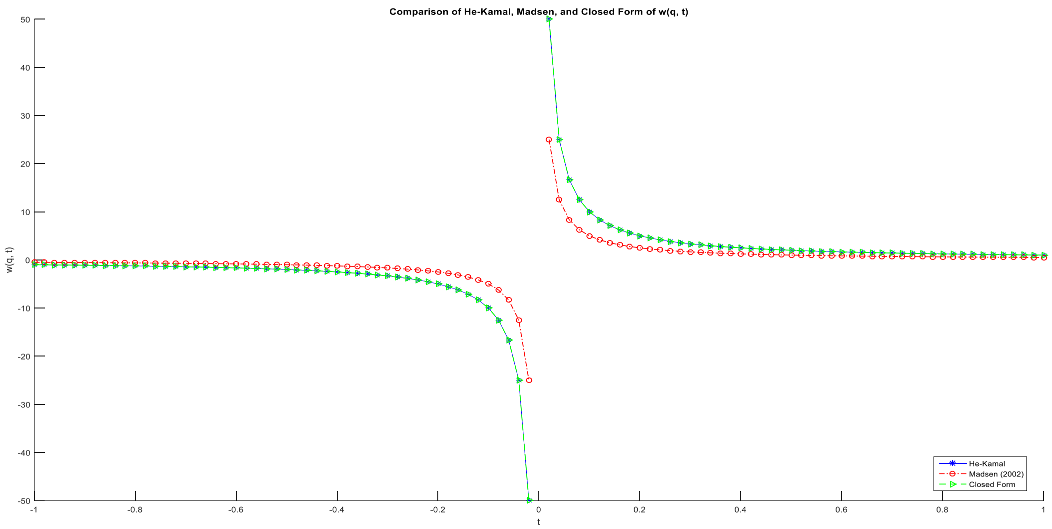

Figure 1: Comparison table of methods with the exact solution and also the error

Table 2: Comparison table of methods with the exact solution and also the error

q	t	HeKamal	Madsen	Exact Solution	Error HeKamal	Error Madsen
-1	-1	-1.0000	-1.0000	-0.5000	0.5000	0.5000
-0.9500	-0.9500	-1.0526	-1.0526	-0.5263	0.5263	0.5263
-0.9000	-0.9000	-1.1111	-1.1111	-0.5556	0.5556	0.5556
-0.8500	-0.8500	-1.1765	-1.1765	-0.5882	0.5882	0.5882
-0.8000	-0.8000	-1.25	-1.25	-0.625	0.625	0.625
-0.7500	-0.7500	-1.3333	-1.3333	-0.6667	0.6667	0.6667
-0.7000	-0.7000	-1.4286	-1.4286	-0.7143	0.7143	0.7143
-0.6500	-0.6500	-1.5385	-1.5385	-0.7692	0.7692	0.7692
-0.6000	-0.6000	-1.6667	-1.6667	-0.8333	0.8333	0.8333
-0.5500	-0.5500	-1.8182	-1.8182	-0.9091	0.9091	0.9091
-0.5000	-0.5000	-2.0000	-2.0000	-1.0000	1.0000	1.0000

Example 3.2. Consider the cubic-Boussinesq equation of the form [14]

$$w_{tt} - (w^3)_{qq} - w_{qqqq} = 0 \qquad (3.17)$$

with

$$w(q,0) = \sqrt{2}\,\mathrm{sech}(q) \qquad w_{0t} = (q,0) = \sqrt{2}\,\mathrm{sech}(q)\tanh(q) \qquad (3.18)$$

Applying the Kamal Transform to (3.17), then, the equation becomes (3.17)

$$\mathcal{K}\left[\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2}{\partial q^2}(w^3) - \frac{\partial^4 w}{\partial q^4}\right] = 0 \quad (3.19$$

Using the recurrence relation (2.12) to equation (3.19) takes the form

$$w_{r+1}(q,v) = w_r(q,v) + \lambda(v)\mathcal{K}\left[\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2}{\partial q^2}(w^3) - \frac{\partial^4 w}{\partial q^4}\right] \quad (3.20)$$

$$\delta w_{r+1}(q,v) = w_r(q,v) + \lambda(v)\left[\left(\frac{1}{v^2}w(q,v) - \frac{1}{v}w(q,0) - w'(q,0)\right) - \mathcal{K}\left\{\frac{\partial^2}{\partial q^2}(w^3) + \frac{\partial^4 w}{\partial q^4}\right\}\right]$$

This implies that

$$\delta w_{r+1}(q,v) = \delta w_r(q,v) + \frac{1}{v^2}\lambda \delta w_r(q,v) \quad (3.22)$$

Consequently,

$$\lambda(v) = -v^2 \quad (3.23)$$

Notice that w_r is a restricted variable, $\delta w_r = 0$

$$\frac{\delta w_{r+1}(q,v)}{\delta w_r(q,v)} = 0 \quad (3.24)$$

The value of $\lambda(v)$ is used in equation (3.23)

$$w_{r+1}(q,v) = w_r(q,v) - v^2\mathcal{K}\left[\frac{\partial^2}{\partial q^2}(w^3) + \frac{\partial^4 w}{\partial q^4}\right] \quad (3.25)$$

Now, taking the inverse Kamal of equation (3.25)

$$w_{r+1}(q,t) = w_r(q,t) - \mathcal{K}^{-1}\left[v^2\mathcal{K}\left[\frac{\partial^2}{\partial q^2}(w^3) + \frac{\partial^4 w}{\partial q^4}\right]\right], \quad (3.26)$$

Applying He's polynomial formula to equation (3.26), we get

$$\begin{aligned} w_0 + \rho w_1 + \rho^2 w_2 + \rho^3 w_3 + \rho^4 w_4 + \cdots \\ = w_r(q,t) - \rho^1\mathcal{K}^{-1}\left[v^2\mathcal{K}\left[\frac{\partial^2}{\partial q^2}({w_0}^3) + \frac{\partial^4 w_0}{\partial q^4}\right]\right] \\ - \rho^2\mathcal{K}^{-1}\left[v^2\mathcal{K}\left[\frac{\partial^2}{\partial q^2}({w_0}^2 w_1) + \frac{\partial^4 w_1}{\partial q^4}\right]\right] \\ - \rho^3\mathcal{K}^{-1}\left[v^2\mathcal{K}\left[\frac{\partial^2}{\partial q^2}(3w_0 w_1^2 + 3w_0^2 w_2) + \frac{\partial^4 w_2}{\partial q^4}\right]\right] \end{aligned}$$

Equating the coefficients of the like powers of ρ yields:

$$\rho^0 : \ w_0 = w_0(q,t) + t w_{0t}(q,t)$$

$$\rho^1 : \ w_1 = -\mathcal{K}^{-1}\left[v^2\mathcal{K}\left[\frac{\partial^2}{\partial q^2}({w_0}^3) + \frac{\partial^4 w_0}{\partial q^4}\right]\right],$$

$$\rho^2 : \ w_2 = -\mathcal{K}^{-1}\left[v^2\mathcal{K}\left[3\frac{\partial^2}{\partial q^2}({w_0}^2 w_1) + \frac{\partial^4 w_0}{\partial q^4}\right]\right],$$

Applying the initial and boundary conditions to obtain w_0

$$w_0(q,0) = \sqrt{2}\,\text{sech}(q) + t(\sqrt{2}\,\text{sech}(q)\tanh(q)), \quad (3.27)$$

$$w_1(q,t) = -\mathcal{K}^{-1}\left[v^2\mathcal{K}\left[\frac{\partial^2}{\partial q^2}({w_0}^3) + \frac{\partial^4 w_0}{\partial q^4}\right]\right]$$

$w_1 = \frac{1}{2}t^2(-\sqrt{2}sech^5(q)\tanh(q) - 4\sqrt{2}sech^3(q)\tanh^3(q) + \sqrt{2}\,\text{sech}(q)\tanh^5(q)) + \frac{1}{6}t^3(-5\sqrt{2}sech^5(q)\tanh(q) - 4\sqrt{2}sech^3(q)\tanh^3(q) + \sqrt{2}\,\text{sech}(q)\tanh^5(q)) + \frac{1}{2}t^4(2\sqrt{2}sech^7(q) - 19\sqrt{2}sech^5(q)\tanh^2(q) + 9\sqrt{2}\,sech^5(q)\tanh^4(q)) + \frac{3}{10}t^5(2\sqrt{2}sech^7(q)\tanh(q) - 9\sqrt{2}sech^5(q)\tanh^3(q) + 3\sqrt{2}\,sech^3(q)\tanh^5(q))$

$$\begin{aligned} w_2(q,t) = \frac{1}{24}t^4(-19\sqrt{2}sech^9(q) \\ &+ 196\sqrt{2}sech^7(q)\tanh^2(q) + 90\sqrt{2}\,sech^5(q)\tanh^4(q)) - 124\,\sqrt{2}sech^3(q)\tanh^6(q) \\ &+ \sqrt{2}\,\text{sech}(q)\tanh^6(q)) + \frac{1}{120}t^5(-11\sqrt{2}sech^9(q)\tanh(q) \\ &+ 316\sqrt{2}sech^7(q)\tanh^3(q) + 162\,\sqrt{2}\,sech^5\,\tanh^5(q)) + \frac{1}{60}t^6(159\sqrt{2}sech^{11}(q) \\ &- 36327\sqrt{2}sech^9(q)\tanh^2(q) + 72819\sqrt{2}\,sech^7(q)\tanh^4(q) \\ &- 23441\sqrt{2}sech^5(q)\tanh^6(q) + 819\sqrt{2}sech^3(q)\tanh^8(q)) \\ &+ \frac{1}{140}t^7(1850\sqrt{2}sech^{11}(q)\tanh(q) \\ &- 11473\sqrt{2}sech^9(q)\tanh^3(q) + 16021\sqrt{2}\,sech^7(q)\tanh^5(q) \\ &- 6023\sqrt{2}sech^5(q)\tanh^7(q) + 273\sqrt{2}sech^3(q)\tanh^9(q)) + \frac{3}{280}t^8(44\sqrt{2}sech^{13}(q) \\ &- 2866\sqrt{2}sech^{11}(q) \\ &+ 14251\sqrt{2}\,sech^9(q)\tanh^4(q) \\ &- 12152\sqrt{2}sech^7(q)\tanh^6(q) + 1575\sqrt{2}sech^5(q)\tanh^8(q)) \\ &+ \frac{3}{40}t^9(4\sqrt{2}sech^{13}(q)\tanh(q) \\ &- 106\sqrt{2}sech^{11}(q)\tanh^3(q) + 357\sqrt{2}\,sech^9(q)\tanh^5(q) \\ &- 236\sqrt{2}sech^7(q)\tanh^7(q) + 25\sqrt{2}sech^5(q)\tanh^9(q)) \end{aligned} \tag{3.29}$$

Hence, the solution of equation (3.17) is expressed as

$$w_r(q,t) = w_0 + w_1 + w_2 + \cdots \tag{3.30}$$

The closed form of equation (3.30) is

$$w(q,t) = \sqrt{2}\,\text{sech}(q-t) \tag{3.31}$$

The result is the same as the result obtained in [14]

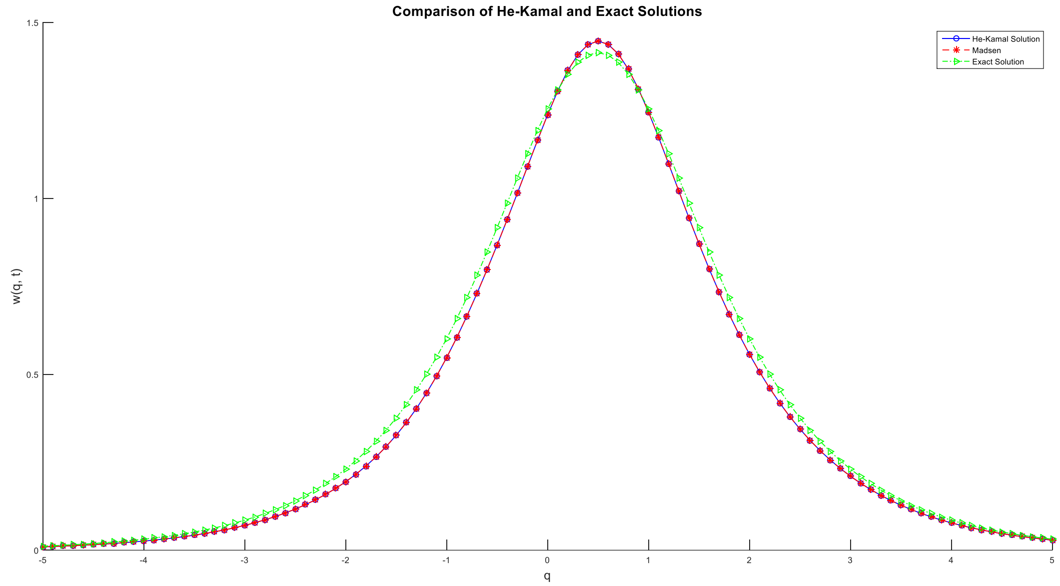

Figure 2: Approximate solution with t=0.5

Table3: Comparison of the methods at t=0.5

q	He_Kamal	Madsen	Exact Solution	He_Kamal Error	Madsen Error
-1	0.5473	0.5473	0.6012	0.0539	0.0539
-0.8	0.665	0.665	0.7175	0.0525	0.0525
-0.6	0.7971	0.7971	0.8476	0.0505	0.0505
-0.4	0.9399	0.9399	0.9868	0.0469	0.0469
-0.2	1.0895	1.0895	1.1267	0.0372	0.0372
0	1.2374	1.2374	1.2542	0.0167	0.0167
0.2	1.3632	1.3632	1.3529	0.0103	0.0103
0.4	1.437	1.437	1.4072	0.0298	0.0298
0.6	1.4378	1.4378	1.4072	0.0306	0.0306
0.8	1.3672	1.3672	1.3529	0.0143	0.0143
1	1.2453	1.2453	1.2542	0.0089	0.0089

4. Conclusion

The paper investigates the numerical solution of the Cubic-Boussinesq problem using a Lagrange multiplier and the He-Kamal transform technique. The equation has difficulties for traditional numerical techniques like HPM, VIM, and integral transform since it contains a cubic nonlinear factor. The original partial differential equation is made simpler using the He-Kamal transform method, which increases its tractability for numerical analysis. A Lagrange multiplier is included to improve the solution's accuracy and converges while successfully handling non-linear components. The wave dynamics and stability characteristics obtained by the He-Kamal transform method with a Lagrange multiplier are clearly seen in MATLAB-generated graphs. The results demonstrate how this combination approach has the potential to further the numerical analysis of nonlinear partial differential equations with increased accuracy and efficiency. This method may be expanded to more complicated systems in future research, and the He-Kamal transform and Lagrange multiplier approaches can be further improved.

5. Acknowledgments
The authors would like to thank the Department of Mathematical Sciences, Olabisi Onabanjo University, Ago Iwoye, Ogun State, Nigeria, which has helped enhance this research's quality.

6. Conflict of Interest
There are no conflicts of interest.

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الحل العددي لمعادلة بوسينسك المكعبة المعدلة باستخدام تحويل هي-كامال مع مضاعف لاغرانج

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الخلاصة
تعتبر معادلة Cubic-Boussinesq ضرورية جدًا لنمذجة ظواهر الموجات غير الخطية، والتقاط ديناميكيات معقدة مثل كسر الموجات وتفاعلات السوليتون. وتكمن أهميتها في قدرتها على وصف سلوك الموجات في سياقات فيزيائية متنوعة، من ديناميات الموائع إلى الألياف الضوئية. نظرًا لعدم الخطية في المعادلة، قد يكون العثور على حلول دقيقة وفعالة أمرًا صعبًا للغاية. تقدم هذه الدراسة طريقة مبتكرة باستخدام طريقة تحويل He-Kamal ومضاعف لاغرانج لحل المعادلة. يعمل تحويل He-Kamal على تبسيط PDE، مما يجعله أكثر قابلية للتتبع، بينما يعزز مضاعف Lagrange دقة الحل وتقاربه. أظهرت المحاكاة العددية أن تحويل He-Kamal بمضاعف لاغرانج يتوافق مع الطرق التقليدية في التعامل مع اللاخطية التكعيبية لمعادلة Cubic-Boussinesq. توضح الرسوم البيانية التي تم إنشاؤها بواسطة MATLAB فعالية الطريقة في التقاط ديناميكيات الموجة واستقرارها. يقدم هذا البحث تقنيات عددية لحل المعادلات التفاضلية الجزئية غير الخطية ويساهم في مجال نظرية الموجات غير الخطية.