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Derivation of Romberg's Rule Using N-Subdivisions and Applications of Romberg's Rule

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Abstract

By using a large number of subdivisions n to obtain good accuracy in approximating the integral $\int_a^b f(x)\,dx$ we note that an increase in the value of n does not necessarily mean an increase in accuracy in the result or even an improvement in the result. In addition, increasing the number n means an increase in the number of times the function values are calculated. The importance of Romberg's rule lies in improving the results obtained using the previous rules and obtaining high accuracy in arriving at approximate results faster and with a small amount of error. The reason for this is the high rank of the Romberg rule compared to the trapezoid rule, for example, which is not accurate in its results because the cutting error for this rule is of the order $O(h^2)$only.

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1. Introduction

Romberg's rule is a numerical integration method that provides a more precise approximation of definite integrals by combining Richardson extrapolation with the trapezoidal rule. It is especially helpful for smooth functions that polynomials can accurately approximate.

Romberg's rule yields a more accurate estimate than the trapezoidal rule by itself when combined with Richardson extrapolation and also Efficiency: Compared to other approaches, it needs fewer function evaluations, particularly for smooth functions. The integral $\int_a^b f(x)\,dx$ may be approximated with excellent precision by using a large number of subdivisions n, which divide the time period into subintervals and yield accurate and correct results. Romberg's rule is important since it yields approximate findings. Compared to earlier techniques that rely on employing fewer divisions, this approach is quicker and has a lesser inaccuracy since it uses high ranks. A useful method for estimating definite integrals, Romberg's rule is a potent numerical integration approach that improves the accuracy of the trapezoidal rule by Richardson extrapolation.

In this research, the following lower triangular matrix of integers is produced using Romberg's method and is all an approximation of the result of the above integral:

$$\begin{matrix} R(1,1) & & \\ R(2,1) & R(2,2) & \\ R(n,1) & R(n,2)\ \ldots & R(n,n) \end{matrix}$$

where R(1,1), R(2,1), and so on for each member of the matrix indicate the first, second, and subsequent approximate values, respectively, using the trapezoid rule.

2. Research Method (Bold, 12 pt)

The following relationships, which rely on the matrix's argument, can be used to locate the elements of the matrix. The trapezoid rule is used to determine the initial value.

$R(1,1) = \frac{b-a}{2}[f(a) + f(b)]$

The first value determines the remainder

$R(k,1) = \frac{1}{2}[R(k-1,1) + h_{k-1} \ \sum_{i=1}^{2^{k-2}} f(a + (i - 0.5))$

$h_{k-1} = \frac{b-a}{2^{k-2}}$, $k = 2,3,\ldots n$

The values obtained from the interpolation are shown in the other columns. The remaining elements in the matrix may be found using the following law:

$R(k,i) = \frac{4^{i-1} R(k,i-1) - R(k-1,i-1)}{4^{i-1}-1}$

$k, i = 2,3,\ldots,n$

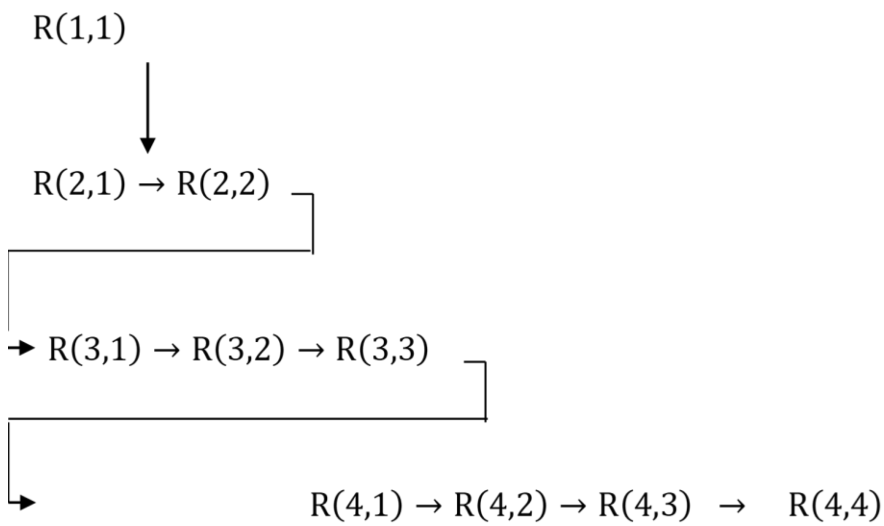

Figure 1. The elements in the matrix Romberg's rule

3. Results And Discussion

3.1 Derivation of Romberg Rule

When the value of the function and its derivatives fall in the interval [a,b], Romperk's method provides a multiple arithmetic approximation of the integral by bisecting h once more.

$$\int_a^b f(x)\,dx \cong \frac{h}{2}\left[f_0 + f_n + 2\sum_{i=1}^{n-1} f_i\right] + \sum_{j=1}^{\infty} \alpha_j h^{2j}$$

where h is the period's duration and αj is the set of all j values independent of h.

Assuming T_(k,0),k=0,1,2,... represents the use of the preceding approximation and $h_k = \frac{b-a}{2^k}$

$I = T_{k,0} + \alpha_1 {h_k}^2 + \alpha_2 {h_k}^4 + \alpha_3 {h_k}^6 + \cdots \quad (1)$

The number of subintervals is now divided in half and doubled.

$I = T_{k+1,0} + \alpha_1(\frac{h_k}{2})^2 + \alpha_2(\frac{h_k}{2})^4 + \alpha_3(\frac{h_k}{2})^6 + \cdots$

$= T_{k+1,0} + \frac{1}{4}\alpha_1 {h_k}^2 + \frac{1}{16}\alpha_2 {h_k}^4 + \frac{1}{64}\alpha_3 {h_k}^6 + \ \ldots (2)$

We may now remove the components that include h^2. To do this, multiply equation (2) by (4) and deduct equation (1) from it. The result is

$3I = 4T_{k+1,0} - T_{k,0} - \frac{3}{4}\alpha_2 {h_k}^4 - \frac{15}{16}\alpha_3 {h_k}^6 + \cdots$

$I = \frac{1}{3}(4T_{k+1,0} - T_{k,0}) - \frac{1}{4}\alpha_2 {h_k}^4 - \frac{5}{16}\alpha_3 {h_k}^6 \ldots (3)$

Where the error value is represented by $\frac{1}{4}\alpha_2 {h_k}^4$ and

$T_{k+1,1} = \frac{4T_{k+1,0} - T_{k,0}}{3}$

It's called completing your first romper.

Equation (3) makes it evident that the value of I is approximated to be of the fourth order, which makes it more accurate (T(k,0),T(k+1,0)) than the second order. As a result, estimations of the sixth order may be obtained by repeating the external interpolation procedure.

$I = T_{k,1} + \beta_1 {h_k}^4 + \beta_2 {h_k}^6 + \cdots \ \ldots (4)$

Since β_1, β_2 is independent on of h, we may derive it by halving h and doubling the subintervals.

$I = T_{k+1,1} + \beta_1 (\frac{h_k}{2})^4 + \beta_2 (\frac{h_k}{2})^6 + \cdots$

$= T_{k+1,1} + \frac{1}{16}\beta_1 {h_k}^4 + \frac{1}{64}\beta_2 {h_k}^6 + \cdots \ \ldots (5)$

After subtracting equation (4) from equation (5) and multiplying it by (4), we obtain

$I = \frac{1}{15}\left(16T_{k+1,1} - T_{k,1}\right) - \frac{1}{20}\beta_1 {h_k}^6$

The second Romberg interpolation, or T(k+1,2), is the prior equation. From there, we proceed to get the best estimate.

Common law

$I = T_{k+1,j+1} + O(h^{2j+4})$

When J=0 Romperk I T(k+1,1)+O(h^4)

When J=1 Romperk II T(k+1,2)+O(h^4)

When J=2 Romperk III T(k+1,3)+O(h^4)

So it is possible to calculate $T_{k+1,j+1} = \frac{4^{j+1} \ T_{k+1,j} - T_{k,j}}{4^{j+1}+1}$

By using Romberg's rule $\int_0^2 x^2 \ dx$, where n=3 we have

R(1,1)

R(2,1) R(2,2)

R(3,1) R(3,2) R(3,3)

$R(1,1) = \frac{b-a}{2}[F(a) + F(b)]$

$= \frac{2-0}{2}[0+4] = 4$

$h_{k-1} = \frac{b-a}{2^{k-2}}$, K=2

$h_1 = \frac{b-a}{2^0} = h_1 = 2$

$h_2 = \frac{2}{2} = 1$

$R(2,1) = \frac{1}{2}[R(1,1) + h_1 \sum_{i=1}^{1} f(0+0.5)(2)]$

$= \frac{1}{2}[4 + 2f(0 + 0.5(2)] = 3$

$R(3,1) = \frac{1}{2}[R(2,1) + h_2 \sum_{i=1}^{2} f(a + (i-0.5)h_2]$

$= \frac{1}{2}\left[3 + f(0+0.5) + f\left(0 + \frac{3}{2}\right)\right]$

$= \frac{1}{2}\left[3 + \frac{1}{4} + \frac{9}{4}\right] = 2.75$

We apply the following formula to get the matrix's remaining members

$R(k,i) = \frac{4^{i-1} \ R(k,i-1) - R(k-1,i-1)}{4^{i-1}-1}$

$R(2,2) = \frac{4R(2,1) - R(1,1)}{4-1}$

$= \frac{4(3)-4}{3}$

$= \frac{8}{3} = 2.66667$

$R(3,2) = \frac{4R(3,1) - R(2,1)}{4-1}$

=2.66667

$R(3,3) = \frac{4^{3-1} \ R(3,2) - R(2,2)}{4^{3-1}-1}$

$= \frac{16(2.66667) - 2.66667}{15}$

=2.66667

This is equivalent to the integral's actual value

$\int_0^2 x^2\ dx = \frac{x^3}{3}\Big|_0^2$

$= \frac{8}{3} \cong 2.6666$

If the following table is used to represent the speed of an automobile at time t, then

T	0	12	24	36	48	60	72	84	96	108	120
V	0	3.60	10.08	18.90	21.60	18.54	10.26	5.30	4.50	5.40	**9.00**

Determine the distance the car travels in a time limit of no more than two minutes using Romberg's rule.

$R(1,1) = \frac{b-a}{2}[f(a) + f(b)]$

$f(a) = f(0) = 0$

$f(b) = f(120) = 120$

$R(1,1) = \frac{120-0}{2}[0 + 120] = 7200$

$R(k,1) = \frac{1}{2}[R(k-1,1) + h_{k-1}\ \sum_{i=1}^{2^{k-2}} f(a + (i - 0.5))$

$h_{k-1} = \frac{b-a}{2^{k-2}}\ ,\ k = 2,3,\ldots..n$

$h_{k-1} = \frac{b-a}{2^{k-2}}\ ,\ k = 2$

$h_{2-1} = \frac{120-0}{2^0} = 120$

$R(2,1) = \frac{1}{2}[R(1,1) + 120\sum_{i=1}^{2^0} f(0 + (i - 0.5)]$

$= \frac{1}{2}[7200 + 120(0.5)] = 216000$

$R(k,i) = \frac{4^{i-1}\ R(k,i-1) - \ R(k-1,i-1)}{4^{i-1}-1}$

$R(2,2) = \frac{4^{2-1}R(2,1) - R(1,1)}{4^{2-1}-1}$

$= \frac{4(216000) - 7200}{3} \approx 280600$

The results are in the form of the following matrix:

7200	**0**
216000	**280600**

Table 1. Romberg's rule is better to the trapezoidal rule because it allows for faster convergence to the integer value.

I \ O(h)	h^2	h^4	h^6	h^8
h	$T_{0,0}$			
$\frac{h}{2}$	$T_{0,1}$	$T_{1,0}$		
$\frac{h}{4}$	$T_{0,2}$	$T_{1,1}$	$T_{2,0}$	
$\frac{h}{8}$	$T_{0,3}$	$T_{1,2}$	$T_{2,1}$	$\mathbf{T_{3,0}}$

3.2 Algorithm of Romberg Rule

Step 1: type f, a, b, and n.

Step 2: We determine the first integral's value $R(1,1) = \frac{b-a}{2}[F(A) + F(B)]$

Step 3: The values of h are determined utilizing $h = \frac{b-a}{(2^{k-2})}$

Step 4: For any value of k and i, we compute the following integral's value.

$R(k,1) = \frac{1}{2}[R(k-1,1) + h_{k-1}\ \sum_{i=1}^{2^{k-2}} f\big(a + (i - 0.5)\big)$

Where $k = 2,3,\ldots,n\ ,\ \ i = 1,2,\ldots,2^{k-2}$

Step 5: We calculate Romperk's rule

$R(k,i) = \frac{4^{i-1}R(k,i-1) - R(k-1,i-1)}{4^{i-1}-1}$

K , i=2,3,…,n

Step 6: Print the integral value

3.3 Program of Romberg Rule

```
f= input('f='); %function  needed to integrate
a= input('a=');  % start point of interval
b= input('b=');  % end point of interval
n= input('n=');
r(1,1)=((b-a)/2)*(subs(f,a)+subs(f,b));
for k=2:n
h=(b-a)/(2^(k-2));
s=0;
for i=1:2^(k-2)
s=s+ subs(f,(a+(i-0.5)*h));
end
r(k ,1)=0.5*(r(k-1,1)+h*s);
end
for i=2:n
for k=i:n
r(k,i)=((4^(i-1)*r(k, i-1) - r(k-1, i-1))/(4^(i-1)-1));
end
r
```

4. Conclusion

The use of a high number of subdivisions, n, in the approximation of the integral is one of the most significant characteristics of using $\int_a^b f(x)\,dx$. Romberg's rule has a significant effect on the outcomes as higher n increases the number of times the function values are computed. As a consequence, it may be applied to mathematical analyses of a variety of mathematical issues and produces fast, highly accurate answers with minimal mistakes, improving the outcomes of the rules before.

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6. References

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اشتقاق قاعدة رومبرك باستخدام التقسيمات الجزئية n وتطبيقات قاعدة رومبرك الرياضية

مصطفى ناظم سالم

فرع العلوم الاساسية، كلية الزراعة والغابات، جامعة الموصل، الموصل، العراق

المستخلص:

باستخدام عدد كبير من التقسيمات الجزئية n لغرض الحصول على دقة جيدة في التقريب للتكامل $\int_a^b f(x)\,dx$ نلاحظ بانه ليس بالضرورة أن يعني الزيادة في قيمة n زيادة في الدقة في النتيجة أو حتى تحسين في النتيجة إضافة إلى ذلك إن زيادة العدد n تعني زيادة بعدد المرات التي تحسب فيها قيم الدالة، إن أهمية قاعدة رومبرك تكمن في تحسين النتائج التي تم الحصول عليها باستخدام القواعد السابقة والحصول على دقة عالية في الوصول إلى نتائج تقريبية بشكل أسرع وبمقدار خطأ بسيط، والسبب في ذلك هي الرتبة العالية لقاعدة رومبرك بالمقارنة مع قاعدة شبه المنحرف مثلاً التي لا تكون دقيقة في نتائجها وذلك لان خطأ القطع لهذه القاعدة من الرتبة $O(h^2)$ فقط.