On GP-Injectivity
With Some Types of Rings

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الملخص

الغرض من هذا البحث هو دراسة المقاسات الغامرة من النمط -GP وأعطاء بعض من هذه الخواص. كذلك برهنا:

1- اذا كان كل مقاس -R أيمن بسيط غامر من النمط -GP و R حلقة قابلة للانعكاس (reversible) ، فأن R حلقة منتظمة ضعيفة من النمط -π

2- لتكن R حلقة منتظمة ضعيفة، حلقة كوازي – ديو اليمنى ، وأن كل مقاس-R أيمن منفرد بسيط غامر من النمط -GP ، فانR حلقة منتظمة من النمط -π .

Abstract

The purpose of this paper is to study GP-injective modules and give some of it is properties. Also, we proved:

(1) If every simple right R-module is GP-injective, and R is reversible ring, then R is a right weakly π-regular.

(2) Let R be a right weakly regular, right quasi-duo ring, if every simple singular right R-module are GP-injective, then R is π-regular.

Introduction

Throughout this paper, R denotes an associative ring with identity and all modules are unitary. J(R), Y(R) and Z(R) are denote the Jacobson radical, the right and left singular ideals of R, respectively. A right R-module M is called Generalized P-injective(briefly, GP-injective) if for any $a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism of $a^n R$ into M extends to one of R into M. This concept was first introduced by Ming [13], Kim, Nam and Kim [9], [10]. An ideal I of a ring R is called reduced if it contains no nonzero nilpotent elements. A ring R is called π-regular [11] if for any $a \in R$, there exists a positive integer n and an element b of R such that $a^n = a^n b\, a^n$. Therefore, R is π-regular if and only if for any $a \in R$, there exists a positive integer n such that $a^n R$ is a direct summand of R. For any nonempty subset X of a ring R, the right (left) annihilator of X will be denoted by r (X) (ℓ(X)), respectively. Recall that:

(1) A ring R is said to be ERT-ring (resp. MERT-ring) [12] if every essential (resp. maximal essential) right ideal of R is two-sided.

(2) Following [14], a ring R is called right (left) quasi-duo if every maximal right (left) ideal of R is two-sided.

(3) R is a right (left) weakly π-regular[6] if, for every a ∈ R, there exists a positive integer n, depending on a such that $a^n \in a^n R a^n R$ ($a^n \in R a^n R a^n$). R is called weakly π-regular if it is both right and left weakly π-regular.

(4) Following [2], a ring R is called a right (left) Ikeda-Nakayama ring(briefly, IN-ring) if it satisfies condition $\ell(I \cap J) = \ell(I) + \ell(J)$, for all right (left) ideals I and J of R.

(5) R is called semi-prime [5] if it contains no nonzero nilpotent ideals.

(6) A ring R is called left non-singular if Z(R) = 0.

(7) An ideal I of a ring R is said to be a nilideal [1] if each element x in I is nilpotent; that is to say, if there exists a positive integer n for which $x^n = 0$, x∈R.

In 1995 [10] and 1999 [9] asked the following questions, respectively.

Question 1:

Is R right weakly π-regular, if every simple right R-module is GP-injective ?

Question 2:

Is R π-regular, if R is a right quasi-duo ring and every simple singular right R-module are GP-injective ?

In this paper, we give answer for these questions and we give condition for every simple right R-module is GP-injective to be right weakly π-regular. As a byproduct, we also show that R is a right quasi-duo ring and every simple singular right R-module are GP-injective is π-regular ring.

2. GP-Injectivity

In this section we discuss the connection between right GP-injectivity with weakly π-regular and π-regular rings.

The following lemma, which is duo to Kim *et. al*. in [10], plays a central role in several of our proofs.

Lemma 2.1:

If R is a right GP-injective ring, then J(R) = Z(R).

Proposition 2.2:

Let R be a right GP-injective and strongly π-regular ring. Then, J(R) is a nilideal.

Proof:

By Lemma 2.1, J(R) = Z(R). Suppose that J(R) is not nilideal. Then, there exists $a \in J(R) = Z(R)$ such that $a^n \neq 0$, for all positive integer n. Since R is strongly π-regular, then $a^n = a^{n+1}b$ for some $b \in R$. Now, let $x \in \ell(ab) \cap Ra^n$. Hence that xab = 0 and $x = ra^n$, for some $r \in R$. So, $0 = ra^n ab = ra^{n+1}b = ra^n = x$. Thus, $\ell(ab) \cap Ra^n = 0$. Then, $a^n = 0$, a contradiction. This proves that J(R) is a nilideal of R. ♦

Recall that a ring R is right (left) GP-injective [3] if for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and $Ra^n = \ell(r(a^n))$ ($a^n R = r(\ell(a^n))$).

Lemma 2.3:

Let R be a semi-prime, ERT and right GP-injective ring. Then, R is a right non-singular.

Proof:

See [7, Theorem 2.3]. ♦

Theorem 2.4:

Let R be a semi-prime, ERT, IN and right GP-injective ring. Then, R is π-regular.

Proof:

Consider a principal left ideal Ra^n for any nonzero $a \in R$, and a positive integer n. Since R is right GP-injective, then $Ra^n = \ell(r(a^n))$. By Lemma 2.3, R is a right non-singular. So, r(a) is not essential right ideal of R. Hence $r(a^n) \oplus T$ is an essential right ideal for some nonzero right ideal T of R. Since R is IN-ring, then $\ell(r(a^n)) + \ell(T) = \ell(r(a^n) \cap T)$ = R, which implies $\ell(r(a^n)) \cap \ell(T) \subseteq \ell(r(a^n) + T) = 0$, since $r(a^n) + T$ is an essential. So, $Ra^n = \ell(r(a^n))$ is a direct summand of R. Therefore, R is π-regular. ♦

Following [4], a ring R is called reversible if ab = 0 implies ba = 0 for $a, b \in R$.

Lemma 2.5: [8]

If R is a reversible ring, then $r(a) = \ell(a)$, for each $a \in R$.

Now, we give under what condition the answer of the first question of a ring is affirmative.

Theorem 2.6:

Let R be a reversible ring. If every simple right R-module is GP-injective, then R is right weakly π-regular.

Proof:

Let $a \in R$ such that $a^2 = 0$ and let M be a maximal right ideal containing r(a). Thus, R/M is GP-injective and so any R-homomorphism of aR into R/M extends to one of R into R/M. Let $f : aR \to R/M$ be defined by $f(ar) = r + M$, for every $r \in R$. Note that f is a well-defined R-homomorphism. Since R/M is GP-injective, there exists $c \in R$ such that $1 + M = f(a) = ca + M$. Now, by Lemma 2.5, $r(a) = \ell(a)$, so $ca \in r(a) \subseteq M$, whence $1 \in M$. This is a contradiction. Therefore, a = 0 and hence R is reduced. We will show that $R y^n R + r(y^n) = R$, for any $y \in R$ and a positive integer n. Suppose that $R y^n R + r(y^n) \neq R$. Then, there exists a maximal right ideal containing $R y^n R + r(y^n)$. Since R/M is GP-injective, there exists a positive integer m such that $y^{nm} \neq 0$ and any R-homomorphism of $y^{nm} R$ into R/M extends to one of R into R/M. Let f : $y^{nm} R \to R/M$ defined by $f(y^{nm} b) = b + M$, for every $b \in R$. Since R is reduced, then f is well-defined R-homomorphism. Since R/M is GP-injective, there exists $c \in R$ such that $1 + M = f(y^{nm}) = c\, y^{nm} + M$, whence $1 - c\, y^{nm} \in M$ and so $1 \in M$. This a contradiction. Therefore, $R y^n R + r(y^n) = R$ and $1 = b + t_1 y^n t_2$, for some $t_1, t_2 \in R$ and $b \in r(y^n)$. Whence R is weakly π-regular. ♦

We now consider other condition for right simple GP-injective to be weakly π-regular.

Theorem 2.7:

Let R be a semi-prime ring with each nonzero right ideal contains a nonzero two-sided ideal. If R is a right simple GP-injective, then it is weakly π-regular.

Proof:

Assume that $0 \neq a \in R$ such that $a^2 = 0$. Then, by assumption there is a nonzero ideal I of R with $I \subseteq aR$. We claim that $\ell(a) \cap I \neq 0$. For if $Ia = 0$, then $I \subseteq \ell(a)$ and if $Ia \neq 0$, then $Ia \subseteq I \cap \ell(a) \neq 0$. Now, $(I \cap \ell(a))^2 \subseteq \ell(a)\, I \subseteq \ell(a)\, aR = 0$. Since R is semi-prime, $I \cap \ell(a) = 0$, which is a contradiction. Consequently, R is reduced. By a similar way of proof is used in previous theorem, we obtain that R is weakly π-regular. ♦

Theorem 2.8:

Let R be MERT, semi-prime ring such that each nonzero right ideal contains a nonzero two-sided ideal. If R is a simple right R-module GP-injective. Then, R is biregular.

Proof:

By a similar way of proof is used in Theorem 2.6, we obtain that R is reduced. For any b $\in$ R, set T = RbR + r(b). Since RbR $\cap$ r(b) = RbR $\cap$ r(RbR) = 0 (because R is reduced), then T = RbR $\oplus$ r(b). Suppose that T $\neq$ R. If M is a maximal right ideal of R containing T, then T is an essential right ideal of R. Since R/T is GP-injective. Now, define $f : b^n R \to R/T$ by $f(b^n r) = r + T$, note that f is a well-defined R-homomorphism. Since R/T is GP-injective, there exists c $\in$ R such that $1 + T = f(b^n) = cb^n + T$ and so $1\text{-}cb^n \in T$, since $b^n \in T$ and R is MERT-ring, this implies that T is a two-sided, and hence $cb^n \in T$. Thus, 1 $\in$ T, a contradiction T $\neq$ R. This proves that RbR = eR, where $e = e^2 \in R$. Since R is reduced, then e is central in R. Whence R is biregular. ♦

3. Rings Whose Simple Singular Modules are GP-injective

In this section the concept of weakly regular rings are considered in connection with simple singular modules are GP-injective and other rings.

We begin this section with the following lemma.

Lemma 3.1:

Every right weakly regular ring is semi-prime.

Proof:

See [10]. ♦

The second question does not hold ingeneral, we consider some conditions under which the answer of it is affirmative.

Theorem 3.2:

Let R be a right weakly regular, right quasi-duo ring. Then, R is π-regular ring, if every simple singular right R-module are GP-injective.

Proof:

Assume that $a^2 = 0$ and a $\neq$ 0. Then, there exist a maximal right ideal M of R such that $a^n R + r(a^n) \subseteq M$. Observe that M must be an essential right ideal of R. If M is not essential, then we can write M = r(e), where $0 \neq e = e^2 \in R$. Since $eR\, a^n R = 0$ and R is semi-prime we have $a^n R\, eR = 0$. Thus, $e \in r(a^n) \subseteq M = r(e)$, where e = 0. It is a contradiction. Therefore, R/M is GP-injective and so any R-homomorphism of $a^n R$ into R/M extends to one of R into R/M. Let $f : a^n R \to R/M$ be defined by $f(a^n x) = x + M$, for all x $\in$ R. Indeed, let $x_1, x_2 \in R$, if $a^n x_1 = a^n x_2$, then $(x_1\text{-} x_2) \in r(a^n) \subseteq M$. Hence $x_1 + M = x_2 + M$. Thus, $f(a^n x_1) = x_1 + M = x_2 + M = f(a^n x_2)$. Therefore, f is a well-defined R-homomorphism.

Since R/M is GP-injective, there exists c $\in$ R such that 1 + M = f (a^n) = c a^n + M. Thus, 1- c $a^n \in$ M, whence 1$\in$ M which is a contradiction. Therefore, R is a reduced and hence a^nR + r(a^n) = R. In particular, 1 = a^nx +b, for some x $\in$ R and b $\in$ r(a^n). So, $a^n = a^n x a^n$. Therefore, R is π-regular. ♦

Corollary 3.3:

Let R be a semi-prime and quasi-duo ring. If every simple singular R-module is GP-injective, then R is a strongly π-regular.

Lemma 3.4 [9]:

If R is a semi-prime ring such that each nonzero right ideal of R contains a nonzero ideal, then R is reduced.

Theorem 3.5:

Let R be a ring such that each nonzero right ideal of R contains a nonzero two-sided ideal. If R is right simple singular GP-injective, then R is a reduced weakly π-regular. ♦

Proof:

Suppose that there exists a nonzero right ideal T of R such that T^n = 0. Then, there exists a nonzero a $\in$ T and a positive integer n such that a^n = 0. First observe that r(a^n) is an essential right ideal of R. If not, there exists a nonzero right ideal K such that r(a^n) $\oplus$ K is right essential in R. Let I be a nonzero ideal of R such that I $\subseteq$ K. Now, a^nI $\subseteq$ a^nR $\cap$ I $\subseteq$ r(a^n) $\cap$ K = 0 and hence I $\subseteq$ r(a^n) $\cap$ K = 0. This is a contradiction. Thus, r(a^n) must be a proper essential right ideal of R. Hence there exists a maximal right ideal M of R containing r(a^n). Obviously M is an essential right ideal of R and R/M is GP-injective. So, any R-homomorphism of a^nR into R/M extends to one of R into R/M. Let f : a^nR $\rightarrow$ R/M be defined by f(a^nr) = r + M, for all r $\in$ R. Then, f is a well-defined R-homomorphism. Since R/M is GP-injective, so there exists c $\in$ R such that 1 + M = f (a^n) = c a^n + M. Now, a^nc $a^n \in a^n$R $a^n \subseteq T^n$ = 0, hence c $a^n \in$ r(a^n) $\subseteq$ M and so 1$\in$ M, which is a contradiction. Therefore, R must be semi-prime. By Lemma 3.4, R is reduced. We will show that R a^nR + r(a^n) = R, for any a $\in$ R and a positive integer n. Suppose that R a^nR + r(a^n) $\neq$ R. Then, there exists a maximal right ideal M of R containing R a^nR + r(a^n). Thus, R/M is GP-injective. So, there exists any R-homomorphism of a^nR into R/M extends to one of R into R/M. Let f : a^nR $\rightarrow$ R/M be defined by f(a^nr) = r + M, for all r $\in$ R. Since R is reduced, then f is a well-defined R-homomorphism. Now, R/M is GP-injective, so there exists c $\in$ R such that 1 + M = f (a^n) = c a^n + M. Hence 1- c $a^n \in$ M and so 1$\in$ M, which is a contradiction. Therefore, R a^nR + r(a^n) = R. In particular, 1 = b a^nd +

x, for some b, d $\in$ R and x $\in$ r(a^n). Therefore, $a^n = a^n b a^n d$. So, $a^n R = a^n R a^n R$ and hence R is weakly π-regular. ♦

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