ON COLOURING MAPS & SEMIDIRECT PRODUCT GROUPOIDS ON GROUP-POSETS

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الملخص

درسنا في هذا البحث الافعال الزمرية اليسرى واليمنى التبادلية على المجموعات المرتبة جزيئا لاعطاء شرط كون الفعل الزمري حرا على التطبيقات اللونية .

كذلك نظرنا الى افعال الزمر على المجموعات المرتبة جزئيا من وجهة نظر نظرية الفصائل وذلك للحصول على بعض النتائج لكروبيئد الجداء شبه المباشر على المجموعات المرتبة جزئيا .

Abstract

In this paper we will study the (G,H)- posets , to give a condition of freeness on colouring maps .

Also we will look at the group – posets from the view point of the category theory . That is to give some results on semi-direct product poset groupoid.

§.0 Introduction :

Much of group theory , particularly that part which deals with finite groups , originated from the study of groups of permutations groups between 1844 and 1900 . Some of the first workers in this area were Lagrange , Galois , and Cauchy [8].

Hence it is natural that the concept of the group actions on sets began as : A group action of finite group G on finite set X to be a group homomorphism □ from G to $S_{/X/}$, the group of permutations on X . The set X is called a G-set [9].

Equivalently , a group G acts on X on the left (X called left G-set) if , to each g∈G and each x ∈ X there corresponds a unique element denoted by ^{g}x in X such that , for all x ∈ X and $g_1,g_2 \in G$:

$$^{g_1}(^{g_2}x) = ^{(g_1 g_2)}(x) \quad \text{and} \quad ^{e}x = x$$

Similarly , we define a right H- set .

In this paper we shall study a left and a right compatible actions on posets and we shall look at the group actions on posets from the viewpoint of category theory .

Finally , a semi-direct product poset groupiod $P \rtimes H$ related to a right H-poset P is considered .

§1 (G,H) – posets :

For any group G and any poset P , we say that G acts on P from the left if to each $g \in G$ and $p \in P$ there corresponds a unique element in P denoted by ${}^{g}p$ such that for all $p,q \in P$ and $g_1,g_2 \in G$:

(i) ${}^{e}p = p$

(ii) ${}^{g_1}({}^{g_2}p) = {}^{(g_1 g_2)}p$

(iii) $p \rangle q$ implies ${}^{g}p \rangle {}^{g}q$

Such poset P with a left action of G on it , is called a left G-poset , or simply a G-poset . If we agnor condition (iii), a G-poset P becomes G-set .

Similarly we can define the right group–posets.

The condition (iii) is different from that which is given in [7] , which is :

(iii) $p \geq q$ implies ${}^{g}p \geq {}^{g}q$

We can conclude that every G-poset P can be considered as a right G-poset (and conversely) which is defined by :

$${}^{g}p = p^{g^{-1}} \text{ for all } p \in P \text{ and } g \in G$$

Definition (1.1) :

A poset P is called (G,H)-poset if P is a left G-poset , a right H-poset and the two actions are compatible , that is each $g \in G$, $h \in H$ and $p \in P$ there corresponds a unique element ${}^{g}p^{h}$ in P such that ${}^{g}p^{h} = {}^{g}(p^{h}) = ({}^{g}p)^{h}$.

In [1] there is another condition on the definition of (G,H)-set, that is ; the right action of H must be regular .

Equivalently, following [10], a poset P is call (G,H)-poset if for all $p \in P$, $g \in G$ and $h \in H$ there exist a unique element ${}^{g}p^{h} \in P$ such that :

(i) ${}^{e}p^{e} = p$

(ii) P is a left G-poset with the action defined by : ${}^{g}p = {}^{g}p^{e}$ for all $p \in P$ and $g \in G$.

(iii) P is a right H-poset with the action defined by : $p^h = {}^e p^h$ for all $p \in P$ and $h \in H$.

(iv) $({}^g p^{\,h}) = ({}^g p)^h = {}^g(p^h)$ for all $p \in P$, $g \in G$ and $h \in H$.

From any group H we will define another group denoted by H^{op} such that $H^{op} = H$ and $h_1 x^{op} h_2$ in H^{op} is $h_2 h_1$ in H for all $h_1, h_2 \in H^{op} = H$.

Lemma (1.2) :

If P is a (G,H)-poset , then it is $(G \times H^{op})$- poset and conversely as the following action :

${}^{(g,h)}p = {}^g p^h$ for all $p \in P$, $g \in G$ and $h \in H$.

Proof :

It is routine chack , following [10 ; proposition (1.15)]. ■

The following theorem gives us an equivalent definition to (G,H)-poset which considers the action as a group homomorphism.

Theorem (1.3) :

(1) let P be a (G,H)-poset , then for every $g \in G$ and $h \in H$ there exist an isomorphism .

${}_g\rho_h : P \rightarrow P$ defined by :

${}_g\rho_h(p) = {}^g p^h$ for all $p \in P$.

Also the map $\rho : (G \times H^{op}) \rightarrow$ Isom (P,P) defined by : $\rho\ (g,h) = {}_g\rho_h$ for all $g \in G$ and $h \in H$ is a homomorphism ,called the (G,H)-action on P.

(2) let P be a poset and a homomorphism $\sigma : (G \times H^{op}) \rightarrow$ Isom (P,P) , then P is a (G,H)-poset with an action defined by : ${}^g p^h = (\sigma(g,h))(p)$ for all $p \in P$, $g \in G$ and $h \in H$.

Proof :

Similar to the proof of theorems (1.1.9) and (1.1.10) in [10]. ■

Lemma (1.4):

Let X and Y be any two posets , then Y^X the set of all the maps from X to Y is a poset with a binary relation defined by :

$\alpha \leq \beta$ if and anly if $\alpha(x) \leq \beta(x)$ for all $x \in X$.

Proof :

The proof is straight forward . ■

Proposition (1.5) :

Let X be a G-poset and Y be a right H-poset , then the poset Y^X is a (G,H)-poset with an action defined by : for any $f \in Y^X$, $g \in G$ and $h \in H$: $({}^g f^h)(x) = (f({}^{g^{-1}}x))^h$ for all $x \in X$

Proof :

(i) Y^X is a G-poset since ;

$$({}^e f)(x) = {}^e f^e(x) = (f({}^{e^{-1}}x))^e = (f({}^e x))^e = (f(x))^e = f(x) \text{ for all } x \in X$$

So, ${}^e f = f$.

$$({}^{g_2}({}^{g_1}f))(x) = ({}^{g_2}({}^{g_1}f^e))^e(x) = (({}^{g_1}f^e)({}^{g_2^{-1}}x))^e = ({}^{g_1}f^e)({}^{g_2^{-1}}x) =$$
$$= (f({}^{g_1^{-1}}({}^{g_2^{-1}}x)))^e(x) = (f({}^{g_1^{-1}g_2^{-1}}x))^e = (f({}^{(g_2g_1)^{-1}}x))^e =$$
$$= ({}^{(g_2g_1)}f^e)(x) = ({}^{(g_2g_1)}f)(x) \text{ for all } x \in X.$$

So, ${}^{g_2}({}^{g_1}f) = {}^{(g_2g_1)}f$.

$$\alpha \le \beta \Rightarrow \alpha(x) \le \beta(x) \text{ for all } x \in X$$
$$\Rightarrow \alpha({}^{g^{-1}}x) \le \beta({}^{g^{-1}}x) \text{ for all } x \in X \text{ and } g \in G$$
$$\Rightarrow (\alpha({}^{g^{-1}}x))^e \le (\beta({}^{g^{-1}}x))^e \Rightarrow ({}^g\alpha^e)(x) \le ({}^g\beta^e)(x)$$
$$\Rightarrow {}^g\alpha^e \le {}^g\beta^e \Rightarrow {}^g\alpha \le {}^g\beta$$

(ii) similarly , Y^X is a right H-poset.

(ii) Now for any $g \in G$, $h \in H$ and $f \in Y^X$;

$$(({}^g f)^h)(x) = (({}^g f)(x)^h) = (f({}^{g^{-1}}x))^h = (f)^h({}^{g^{-1}}x) =$$
$$= ({}^g(f^h))(x) \text{ for all } x \in X.$$

So, $({}^g f)^h = {}^g(f)^h$.

Therefore Y^X is a (G,H)-poset. ■

§.2 : Freeness on colouring maps:

Let X be a G- poset and a poset C be a set of colours which is a right H- poset .

In this section we shall give the conditions of freeness on C^X, the set of colouring maps from X to C .

Definition (2.1):

A G- poset X is called free G-poet if :

$Stab_G(x) = \{g \in G : {}^g x = x\} = \{e\}$ for all $x \in X$. [5]

Similarly , the definition of free right H-poset and free (G,H) – poset .

In genral , if a (G,H)- set X is free , then X is a free G-set and a free right H-set. But the converse is not true. [10]

In the following propositions we shall prove that a (G,H)-poset C^X is free if and only if it is free G-poset with G = {e} and the right H-poset C is free.

Proposition (2.2):

The right H-poset C is free if and only if C^X is a free right H-poset .

Proof :

Let C be a free right H-poset.

Let $\alpha \in C^X$ such that $\alpha^h = \alpha$ for some $h \in H$.

Hence $(\alpha^h)(x) = \alpha(x)$ for all $x \in X$.

So, $(\alpha(x))^h = \alpha(x)$ for all $x \in X$.

Since C is a free right H-poset , then h = e .

Therefore C^X is a free right H-poset .

Conversely , suppose that C^X is a free right H-poset.

Let $c \in C$ and $c^h = c$ for some $h \in H$.

Let $\alpha \in C^X$ defined by $\alpha(x)=c$ for all $x \in X$.

Hence $(\alpha^h)(x) = (\alpha(x))^h = c^h = c = \alpha(x)$. So $\alpha^h = \alpha$. But C^X is a free right H-poset , so h = e .

Therefore C is a free right H-poset. ■

Proposition (2.3):

The G-poset C^X is free if and only if G = {e}.

Proof :

It is obvious that C^X is free G-poset when G= {e}.

Now suppose that C^X is a free G-poset.

Let $c \in C$ and consider the map $\alpha \in C^X$ defined by :

$\alpha(x)=c$ for all $x \in X$. So, $({}^g\alpha)(x) = \alpha({}^{g^{-1}}x) = c = \alpha(x)$ for all $x \in X$. Hence ${}^g\alpha = \alpha$ for all $g \in G$.

Now since C^X is a free G-poset . Therefore G= {e}. ■

Proposition (2.4):

The (G,H)- poset C^X is free if and only if G = {e} and C is a free right H- poset.

Proof :

Suppose that $C^X$is a free (G,H)- poset.

Let $\alpha \in C^X$ with ${}^g\alpha = \alpha$ for some $g \in G$.

Then ${}^g\alpha^e = {}^g\alpha = \alpha$. So, g=e , that is since C^X is a free (G,H)-poset. Hence C^X is a free G-poset and by [prop(2.3)] G = {e}.

Similarly we prove that C^X is a free right H-poset.

Hence , by [prop. (2.2)] , C is a free right H-poset.

Conversely , suppose that G = {e} and C is a free right H-poset.
So, C^X is a free G- poset since G = {e}.
Now let $\alpha \in C^X$ and ${}^g\alpha^h = \alpha$ for some $g \in G$ and $h \in H$.
Since G = {e} , $\alpha^h = \alpha$. Hence h = e.
Therefore C^X is a free (G,H)- poset. ■

§.3 G- posets as G- objects of the category Posets

In this section we look at group – **P**osets from the view point of category theory.

Proposition (3.1) :

Every poset is a category .

Proof :

Let P be a poset .Then we consider P to be the category **P** with :
(i) ob (**P**) = P . (ii) for any a,b ∈ ob (**P**) ; there exist a unique arrow denoted (a,b) if and only if a ≤ b . (iii) For any arrows (a , b) , (b , c) , (a , b) (b , c) = (a , c) , since (a ≤ b , b ≤ c) implies a ≤ c.
(iv) For every object a of **P** , the arrow (a , a) denoted by 1_a is an identity , since 1_a(a , b) = (a , b) and (x , a)1_a = (x , a).

Remarke (3.2):

For every poset P , $|$**P** (x,y)$|$= 1 or zero.
Hence , when x ≠ y ∈ P , P (x,y) ≠ ϕ implies . **P** (y,x) = ϕ , but the converse is not true .

Proposition (3.3):

The set of all the posets is a category denoted by **P**osets.

Proof :

(i) ob (**P**osets) is the set of all posets .
(ii) If P,Q ∈ ob (**P**osets) , then f ∈ (**P**osets) (P,Q) if and only if f ∈ Hom (P,Q) . (iii) If f ∈ **P**osets (P,Q) and g ∈ **P**osets (Q,W) , then fg = gof. (iv) for any P∈ **P**osets , 1p is the identity map on P. ■

Let (G , *) be a finite group , then **G** is a category with a single object G and the arrows are the elements of G . So we can consider a functor F from **G** to **P**osets . Hence F(G) is an object in **P**osets , a poset P say . For each arrow g of G the image F(g) : P →P is a poset isomorphism with $(F(g))^{-1} = F(g^{-1})$. So {F(g): g ∈ Hom (G,G)} is a subgroup of the group of isom (P,P). Therefore , F determines a G- action on the poset P.

For more generalization , we can define a group action on poset a s group action on object of **P**osets . For more details see [4] , [6], [7].

Definition (3.4) :

A left G- object in the category **P**osets is a pair (α , P) with P is an object of **P**osets and α a left action on P defined by ; to each $g \in G$, there corresponds a unique morphism $\alpha_g \in$ Isom (P,P) such that :

$$\text{(i) } \alpha_{g_1 g_2} = \alpha_{g_1} \circ \alpha_{g_2} \text{ for all } g_1, g_2 \in G$$

(ii) $\alpha_e = 1_P$ where $1_P : P \rightarrow P$ is the identity map.

Proposition (3.5):

The definition of G- object (α , P) in the category **P**osets is equivalent to the definition of a G- poset P.

Proof :

Let (α , P) be a G-object in the category **P**osets .For each $p \in P$ and $g \in G$ let ${}^g p = \alpha_g(p)$.

Hence ; (i) ${}^e p = \alpha_e(p) = 1_P(p) = p$.

$$\text{(ii) } {}^{(g_1, g_2)}p = \alpha_{g_1 g_2}(p) = (\alpha_{g_1} \circ \alpha_{g_2})(p) = \alpha_{g_1}(\alpha_{g_2}(p)) = {}^{g_1}({}^{g_2}p), \text{ for all } g_1, g_2 \in G$$

(iii) For $p,q \in P$, if $p > q$ then $\alpha_g(p) > \alpha_g(q)$, that is , ${}^g p > {}^g q$.

Conversely ; suppose that P is a G-poset . Then there exists a homomorphism $\alpha : G \rightarrow$ Isom (P,P) such that :

$\alpha(g) = \alpha_g \in$ Isom (P,P) with ;

$(\alpha(g))(p) = {}^g p$ for all $p \in P$. So ;

$$\text{(i) } \alpha_{g_1 g_2}(p) = {}^{(g_1 g_2)}(p) = {}^{g_1}({}^{g_2}p) = \alpha_{g_1}(\alpha_{g_1}(p)) =$$
$$= (\alpha_{g_1} \circ \alpha_{g_2})(p) \text{ for all } p \in P \text{ and } g_1, g_2 \in P$$

(ii) $\alpha_e(p) = {}^e p = p = 1_P(p)$ for all $p \in P$

Therefore ; (α , P) is a G-object in the category **P**osets. ■

§.4 Semi- direct product poset groupoid :

A groupoid should be thought of as a group with many objects , or with many identities .

The definition of groupoid were introduced by Brands in his 1929 paper [2] , is given with extra condition on the definition bellow , such a groupoid we nowadays called connected or transitive . [3]

Definition (4.1):

A groupoid **G** is a small category in which every arrow is invertible (every morphism is an isomorphism) .

Hence **G** has a set of morphims , may be called elements of **G** , a set ob(**G**) of objects , together with maps s,t : **G**→ob(**G**), i:ob(**G**) →**G** such that si = ti = 1 the identity map . The maps s,t called the source and target maps respectively .

If x , y ∈ **G** and t(x) = s(y) , then a product xy is exists such that s(xy) = s(x) , t(xy) = t(y) , and this product is associative . For every a ∈ ob(**G**) , the element i(a) is the identity morphism of a .

Also each element x has an inverse x^{-1} with $s(x^{-1}) = t(x)$, $t(x^{-1}) = s(x)$, $xx^{-1} = (is)(x)$, $x^{-1}x = (it) = (x)$.

Hence we can consider any group G to be a grouoid **G** with one object G and the arrows are the elements of G. So the definition of groupoids is an extension of that of groups.

The basic reference for the theory of groupoids is Higgins' book [4] , a survy of the wide range of applications of the theory is given by Brown in [3].

Example [4.2] :

An equivalence relation R on a set X becomes a groupoid with X is the set of the objects , R is the set of arrows and product :

(x,y) (y,z) = (x,z) whenever (x,y) , (y,z) ∈ R.

This example is due to croisot [3] . A special case of this groupoid is the coarse groupoid $X\bar{x}X$, which obtained by taking R = XxX .

Lemma (4.3):

Let P be a right H-poset . Then PxH is a right H-poset.

Proof :

1. PxH is a poset defined by (p,a) ≥ (q,b) if and only if p ≥ q.
2. PxH is a right H-poset with action defined by :

$(p,a)^h = (p^h,a)$ for all p∈P , a , h ∈ H ; that is since :

(i) $(p , a)^e = (p^e, a) = (p,a)$

(ii) $((p,a)^{h_1})^{h_2} = (p^{h_1}, a)^{h_2} = ((p^{h_1})^{h_2}, a) = (p^{h_1 h_2}, a) = (p,a)^{h_1 h_2}$

(iii) $(p,a) > (q,b) \Rightarrow p > q \Rightarrow p^h > q^h \Rightarrow (p^h,a) > (q^h,a)$
$\Rightarrow (p,a)^h > (q,b)^h$ ■

Proposition (4.4) :

Let P be a right H-poset . Then we have a groupoid denoted by $P \rtimes H$ with P is the object set and arrows (p,h) : p →$p^{h^{-1}}$ with p ∈ P and h ∈H , and the product :

$(p,h)\ (p^{h^{-1}},t) = (p, th)$ with p ∈ P , h,t ∈ H .

Proof :

1. For any $p \in P$ and $h \in H$ there exists a unique arrow $(p,h) : p \to p^{h^{-1}}$.

So, $p \xrightarrow{(p,h)} p^{h^{-1}} \xrightarrow{(p^{h^{-1}},t)} (p^{h^{-1}})^{t^{-1}} = p^{(th)^{-1}}$

i.e : $(p,h)\,(p^{h^{-1}},t) : p \to p^{th^{-1}}$. But $(p , th) : p \to p^{(th)^{-1}}$.

Hence $(p,h)\,(p^{th^{-1}},t) = (p, th)$.

2. For $p \in P$, $1_p = (p,e) : p \to p^{e^{-1}} = p^e = p$.

So , $(p,e)(p,t) = (p,e)\,(p^e,t) = (p,e)\,(p^{e^{-1}},t)$

$= (p, te) = (p,t)$ for all $t \in H$.

Similarly $(p,t)\,(p,e) = (p,t)$ for all $p \in P$, $t \in H$.

3.For any arrow $(p,h):p \to p^{h^{-1}}$,$(p,h)(p^{h^{-1}}, h^{-1}) = (p, h^{-1}h) = (p,e) = 1_p$.

Similarly $(p^{h^{-1}},h^{-1})\,(p,h) = (p^{h^{-1}}, h^{-1})\,((p^{h^{-1}})^h,h)$

$= (p^{h^{-1}}, hh^{-1}) = (p^{h^{-1}},e) = 1_{(p^{h^{-1}})}$.

Therefore the arrow (p,h) is invertible and it's inverse is the arrow $(p^{h^{-1}}, h^{-1})$. ■

The groupoid $P \rtimes H$ may be called semi-direct product poset groupoid because this groupoid is a special case of the semi-direct product groupoid obtained from an action of a group on a set . [8]

Note that for any right H- poset P we have two groupoids , $P\bar{x}P$ considering P as a set and $P \rtimes H$, we shall prove that $P\bar{x}P \cong P \rtimes H$ if and only if P is a regular right H – poset .

Proposition (4.5):

Let $F : P \rtimes H \to P\bar{x}P$ defined by ; $F(p) = p$ and $F(p,h) = (p,p^{h^{-1}})$ for all $p \in P$ and $h \in H$. Then F is a functor .

Proof :

$F(1_p) = F(p,e) = (p,p^{e^{-1}}) = (p,p) = 1_p = 1_{F(p)}$

$F((p,h)(p^{e^{-1}},t)) = F(p,th) = (p,p^{(th)^{-1}})$

$(F(p,h))(F(p^{h^{-1}},t)) = (p,p^{h^{-1}})(p^{h^{-1}},p^{(th)^{-1}}) = (p,p^{(th)^{-1}})$

Hence ; $F((p,h)(p^{h^{-1}},t)) = (F(p,h))\,(F(p^{h^{-1}},t))$. ■

Definition (4.6) :[5]

A right H- poset P is called right transitive H- poset if $P \neq \phi$ and for any p , q $\in$ P there exists $h \in H$ such that $p^h = q$.

Equivalently , if there is some element $p \in P$ such that for any $q \in P$ there exists $h \in H$ with $p^h = q$.

Definition (4.7) :[5]

A right H- poset P is called right regular H-poset if it's free and transitive right H- poset .

Proposition (4.8) :

$P \rtimes H \cong \overline{PxP}$ by the functor F defined above if and only if the right H- poset P is regular .

Proof :

Suppose that $P \rtimes H \cong \overline{PxP}$ by the functor F. Then F is an isomorphism functor . That is F is onto and 1-1.

Now let $p,q \in P$. Since F is onto then there exists $h \in H$ such that $F(p,h) = (p,q)$. So , $(p,p^{h^{-1}}) = (p,q)$.

Hence $q = p^{h^{-1}}$. So P is a transitive right H-poset .

Let $p^h = p$ and $k = h^{-1}$ .Then $F(p,e) = (p,p^{e^{-1}}) = (p,p)$ and $F(p,k) = (p,p^{k^{-1}}) = (p,p^h) = (p,p)$. So $F(p,e) = F(p,k)$. Since F is injective , $(p,e) = (p,k)$. That is $k = e$. Hence $h = e$ and the right H-action on P is free.

Therefore the right H- poset P is regular .

Conversely , suppose that P is regular right H-poset . Then P is transitive and free right H-poset .

Let $(p,q) \in PxP$. Since P is transitive right H-poset, then there exists $h \in H$ such that $q = p^h$.

So, $(p,h^{-1}) \in (P \rtimes H)$ and $F(p,h^{-1}) = (p,p^h) = (p,q)$

Also , let $p \in P$ then $F(p) = p$ .Hence F is onto .

Now let $F(p,h) = F(q,k)$ .Then $(p,p^{h^{-1}}) = (q,q^{h^{-1}})$

So, $p = q$ and $p^{h^{-1}} = q^{k^{-1}}$. Hence $p^{h^{-1}k} = q$.

Since the right H- poset P is free, $h^{-1}k = e$. So , $h=k$.

Also , let $F(p) = F(q)$ then $p = q$ .Hence F is injective .

Therefore F is an isomorphism and $P \rtimes H \cong \overline{PxP}$. ■

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