UPPER AND LOWER BOUNDS OF THE BASIS NUMBER OF KRONECKER PRODUCT OF A WHEEL WITH A PATH AND A CYCLE

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الملخص

يعرف العدد الأساس *b(G)* لبيان *G* على انه اصغر عدد صحيح موجب *k* بحيث ان *G*ـلـ قاعدة ذات ثنية *k* لفضاء دارته. في هذا البحث سوف ندرس القيد الاعلى والاصغرللعدد الأساس لجداء Kronecker للعجلة مع الدرب والدارة حيث توصلنا إلى النتائج الأتية:

$3 \le b(W_m \otimes P_n) \le 4$, $m \ge 4$ and $n \ge 3$,
$3 \le b(W_m \otimes C_n) \le 5$, $m \ge 4$, $n \ge 3$.

ABSTRACT

The basis number, $b(G)$,of a graph G is defined to be the smallest positive integer k such that G has a k-fold basis for its cycle space. We investigate upper and lower bounds of the basis number of Kronecker product of a wheel with a path and a cycle. It is proved that

$3 \le b(W_m \otimes P_n) \le 4$, $m \ge 4$ and $n \ge 3$,
and
$3 \le b(W_m \otimes C_n) \le 5$, $m \ge 4$, $n \ge 3$.

1. INTRODUCTION.

Throughout this paper, we consider only finite, undirected and simple graphs. Our terminology and notations will be standard except as indicated. For undefined terms, see [3] .

Let G be a connected graph, and let $e_1, e_2, \ldots, e_q$ be an ordering of the edges. Then any subset S of edges corresponds to a (0,1)-vector $(a_1, a_2, \ldots, a_q)$ in the usual way, with $a_i =1$ if $e_i \in S$ and $a_i =0$ otherwise, for $i=1,2,\ldots q$. These vectors form a q-dimensional vector space, denoted by $(Z_2)^q$ over the field Z_2.

The vectors in $(Z_2)^q$ which correspond to the cycles in G generate a subspace called the cycle space of G, and denoted by $\xi(G)$. It is well known that

$$\dim \xi(G)= \gamma(G)=q-p+k,$$

where p is the number of vertices, k is the number of connected components and $\gamma(G)$ is the cyclomatic number of G. A basis for $\xi(G)$ is called h-fold if each edge of G occurs in at most h of the cycles in the basis. The basis number of G, denoted by $b(G)$, is the smallest positive integer h such that $\xi(G)$ has an h-fold basis, and such a basis is called a required basis of G and denoted by $B_r(G)$. If B is a basis for $\xi(G)$ and e is an edge of G, then the fold of e in B, denoted by $f_B(e)$ is defined to be the number of cycles in B containing e.

Definition: Let $G=(V,E)$ be a simple graph with order n and vertex set $V=\{p_1,p_2,\ldots, p_n\}$. the adjacency matrix of G, denoted by $A(G)$ is the $n\times n$ matrix defined by :

$$A(G)=[a_{ij}]_{n\times n} \text{ where } a_{ij} =\begin{cases}1 & \text{,if the edge } p_ip_j \text{ in E ,}\\ 0 & \text{, otherwise}\end{cases}$$

a_{ij} is called the adjacency number of the pair (v_i, v_j) of vertices.

Definition: Let the vertex sets of the graphs G and H be $\{p_i \mid i=1,2,\ldots,m\}$ and $\{q_j \mid j=1,2,\ldots,n\}$ *resp.*,then the Kronecker product [8], $G\otimes H$,is the graph with vertex set $\{(p_i,q_j): \text{ for } i=1,2,\ldots,m \text{ and } j=1,2,\ldots,n\}$ such that the adjacency number of the pair $(p_i,q_j), (p_k,q_\ell)$ is the product of the adjacency numbers of (p_i,p_k) in G and (q_j,q_ℓ) in H. $G\otimes H$ is also called direct product(tenser product) of G and H,and may be denoted by G.H [1]. $G\otimes H$ is also defined as

$V(G\otimes H)=V(G)\text{x}V(H)$

$E(G\otimes H)=\{(p_i,q_j)(p_k,q_\ell) \mid p_i p_k \in E(G) \text{ and } q_j q_l \in E(H)\}$.

The Kronecker product is commutative (up to isomorphism) and associative [7] .

The first important result of the basis number occured in 1937 when MacLane [5] proved that a graph G is planar if and only if $b(G)\leq 2$. In 1981, Schmeichel [6] proved that for $n\geq 5, b(K_n)=3$, and for $m,n \geq 5$, $b(K_{m,n})=4$ except for $K_{6,10}$, $K_{5,n}$ and $K_{6,n}$ in which n=5,6,7 and 8.
Moreover, in 1982, Banks and Schmeichel [2] proved that for $n\geq 7$, $b(Q_n)=4$, where Q_n is the n-cube.
The purpose of this paper is to determine upper and lower bounds of the basis number of Kronecker product of a wheel with a path and a cycle.

2.1. On the Basis Number Of $W_m \otimes P_n$.

In this section, we obtain upper and lower bounds for the basis number of kronecker product of a wheel with a path.Let the vertex sets of C_m and P_n be Z_m and Z_n respectively, where Z_n denotes the additive group of residues modulo n. Let the cycle C_m be $0,1,2,\ldots,m-1,0$.
The following lemma is needed in the proof of the following theorem which is due to Weichsel [8].

Lemma1:If G and H are connected graphs then the Kronecker product $G \otimes H$ is connected if and only if either G or H contains an odd cycle.
Let W_m be the join of a cycle $0\ 1\ 2\ \cdots (m-2)\ 0$ with the vertex α and let $P_n = 0\ 1\ 2 \cdots (n-1)$.(See Fig.1).

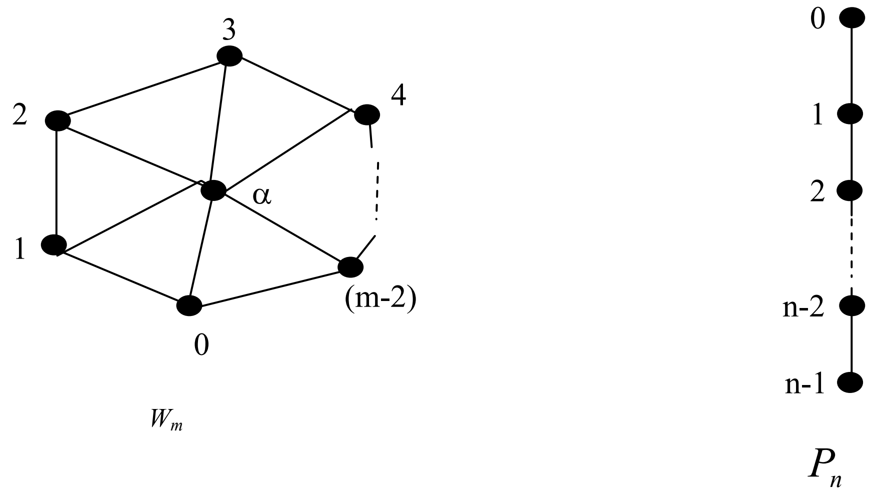
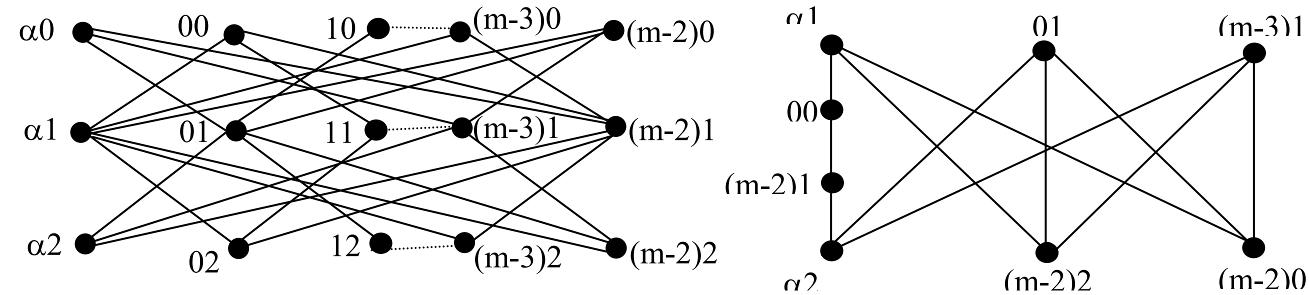


Fig.1

Theorem 2. For $m \geq 4$ and $n \geq 3$, $3 \leq b(W_m \otimes P_n) \leq 4$.

<u>**Proof:**</u> One can easily observe from Fig.2, that $W_m \otimes P_3$ contains a subgraph H homeomorphic to $K_{3,3}$. Thus by Kuratowskis theorem [3], $W_m \otimes P_3$ is non planar, since $W_m \otimes P_3$ is a subgraph of $W_m \otimes P_n$ for $n \geq 3$; then by MacLanes theorem [5], $b(W_m \otimes P_n) \geq 3$.



$W_m \otimes P_3$ H

Fig.2: $W_m \otimes P_3$

To complete the proof we find a 4-fold basis $B(W_m \otimes P_n)$ for $\xi(W_m \otimes P_n)$. We have two cases:

Case (1): " *m"* is even. Let

$B = B(C_{m-1} \otimes P_n) \cup N \cup P \cup M$, .

where $B(C_{m-1} \otimes P_n)$ is the basis for $\xi(C_{m-1} \otimes P_n)$ discussed in Theorem 2.2.1[4] in which "*m-1*" is odd, that is,

$B(C_{m-1} \otimes P_n) = \{ij, (i-1)(j+1), i(j+2), (i+1)(j+1), ij : i \in Z_{m-1} \quad and \quad j = 0,1,\ldots,n-3\} \cup \{S\}$,

where $S = 00,11,20,31,\ldots,(m-2)0,01,10,21,30,\ldots,(m-2)1,00$,

$N = \{ij, (i+1)(j+1), \alpha j, i(j+1), (i+1)j, \alpha(j+1), ij : i = 0,1,2,\ldots,m-3 \quad and \quad j = 0,1,2,\ldots,n-2\}$,

$P = \{(m-2)j, 0(j+1), \alpha j, (m-2)(j+1), 0j, \alpha(j+1), (m-2)j : j = 0,1,\ldots,n-3\}$ and

$M = \{\alpha j, i(j+1), (i+1)j, (i+2)(j+1), \alpha j$

$and \ \alpha(j+1), ij, (i+1)(j+1), (i+2)j, \alpha(j+1) : i = 0,2,4,\ldots,m-4$

$and \ \ j = 0,1,\ldots,n-2\}$.

It is clear that

$|B| = (m-1)(n-2) + 1 + (m-2)(n-1) + (n-2) + (m-2)(n-1)$

$= 3mn - 4m - 4n + 5 = \gamma(W_m \otimes P_n)$.

We shall prove that B is independent.

First, the cycles of $N \cup P$ and M are independent for each $j = 0,1,\ldots,n-2$ since any linear combination of cycles in $N \cup P$ or M for some $i = 0,1,\ldots,m-2$ contains edges of the form, $ij \ ,(i+1)(j+1) \quad or \quad i(j+1) \ ,(i+1)j$.

That is, any linear combination of cycles in $N \cup P$ and M is not equal to zero modulo (2). Moreover, for all $j = 0,1,\ldots,n-2$, every cycle of $N \cup P$ contains an edge of the form $\alpha j \ ,i(j+1) \ or \ \alpha(j+1) \ ,ij$ for some

$i = 1,3,5,...,m-2$ which is not present in any cycle of *M.* Also the cycles in $N \cup P \cup M$ satisfy $(N_j \cup P_j \cup M_j) \cap (N_k \cup P_k \cup M_k) = \Phi$ for all $j \neq k$ where N_j is defined as follows:

It is clear that the vertex set of $W_m \otimes P_n$ can be partitioned into $V_0, V_1, ..., V_{n-1}$, where

$V_j = \{(i,j) : i = \alpha, 0,1,2,...,m-2\}$.

Notice that $V(W_m) = \{\alpha, 0,1,2,...,m-2\}$.

Now, N_j is the cycle of *N* that join a vertex of V_j to a vertex of V_{j+1}, for each $j = 0,1,...,n-2$.

By a similar method, we define P_j and M_j .

Moreover, for every nonconsective integers j and k in *{0,1,…,n-2}*, every cycle in $N_j \cup P_j \cup M_j$ is edge-disjoint with every cycle in $N_k \cup P_k \cup M_k$. Furthermore, if C_i is any cycle in $N_j \cup P_j \cup M_j$, $j = 0,1,...,n-3$ then C_i contains the edge $ij\ ,(i+1)(j+1)$ which is not contained in any cycle in $N_{j+1} \cup P_{j+1} \cup M_{j+1}$. This shows that $N \cup P \cup M$ is independent. Moreover, the cycles of $N_j \cup P_j \cup M_j$ for all $j = 0,1,...,n-2$ are independent from the cycle of $B(C_{m-1} \otimes P_n)$ because if C'_i is any cycle generated from cycles in $N \cup P \cup M$, then C'_i contains an edge of the form $\alpha j\ ,i(j+1)$ *or* $\alpha(j+1)\ ,ij$ for some $i = 0,1,...,m-2$ which is not present in any cycle of $B(C_{m-1} \otimes P_n)$. Thus $B(W_m \otimes P_n)$ is independent set of cycles and so it is a basis for $\xi(W_m \otimes P_n)$.

We now consider the fold of $B(W_m \otimes P_n)$. Partition, the edge set of $W_m \otimes P_n$ into $ij\ ,(i+1)(j+1)$ *or* $i(j+1)\ ,(i+1)j$ and $\alpha j\ ,i(j+1)$ *or* $\alpha(j+1)\ ,ij$ in which $i \in Z_{m-1}$ and $j = 0,1,...,n-2$. Thus if *e* is any edge in $W_m \otimes P_n$ of the form $ij\ ,(i+1)(j+1)$ *or* $i(j+1)\ ,(i+1)j$, then

$$f_{B(C_{m-1} \otimes P_n)}(e) \leq 2, \quad f_{N \cup P}(e) \leq 1, \quad f_{M}(e) \leq 1$$

and so

$$f_{B(W_m \otimes P_n)}(e) \leq 4 .$$

While, if *e* is any edge in $W_m \otimes P_n$ of the form $\alpha j\ ,i(j+1)$ *or* $\alpha(j+1)\ ,ij$,then

$$f_{B(C_{m-1} \otimes P_n)}(e) = 0, \quad f_{N \cup P}(e) \leq 2, \quad f_{M}(e) \leq 2$$

and so

$f \qquad (e) \leq 4$.

$B(W_m \otimes P_n)$

Therefore, $B(W_m \otimes P_n)$ is a 4-fold basis.

<u>Case (2):</u> " m" is odd. Let

$B(W_m \otimes P_n) = B^*(C_{m-1} \otimes P_n) \cup M^* \cup N$.

Where $B^*(C_{m-1} \otimes P_n)$ is a basis for $\xi(C_{m-1} \otimes P_n)$ discussed in[1,Theorem 1, case(2)] namely,

$$B^*(C_{m-1} \otimes P_n) = \{ij, (i+1)(j-1), (i+2)j, (i+1)(j+1), ij$$
$$: i \in Z_{m-1}, j = 1,2,...,n-2 \ and \ (j-i) \ is \ \ even\} \cup$$
$$\{ij, (i+1)(j-1), (i+2)j, (i+1)(j+1), ij :$$
$$i \in Z_{m-1}, \ j = 1,2,...,n-2 \ and \ (j-i) \ is \ \ odd\},$$

$$M^* = \{\alpha j, i(j+1), (i+1)j, (i+2)(j+1), \alpha j \ and \ \alpha(j+1), ij, (i+1)(j+1), (i+2)j, \alpha(j+1)$$
$$: i = 0,2,4,...m-3 \ \ \mathrm{mod}(m-1) \ and \ \ j = 0,1,2,...,n-2\}$$

and N is same as Case (1).

It is clear that

$$|B(W_m \otimes P_n)| = (m-1)(n-2) + (m-1)(n-1) + (m-2)(n-1)$$
$$= mn - 2m - n + 2 + mn - m - n + 1 + mn - m - 2n + 2$$
$$= 3mn - 4m - 4n + 5 = \gamma(W_m \otimes P_n) \ .$$

As in the proof of Case (1), we can show that $B(W_m \otimes P_n)$ is independent set of cycles and so it is a basis for $\xi(W_m \otimes P_n)$ of fold 4,that is $B^* = B(W_m \otimes P_n)$.

Note that if " m" is even and $m \geq 4$,then $W_m \otimes P_2$ is planar graph,(see Fig.3).

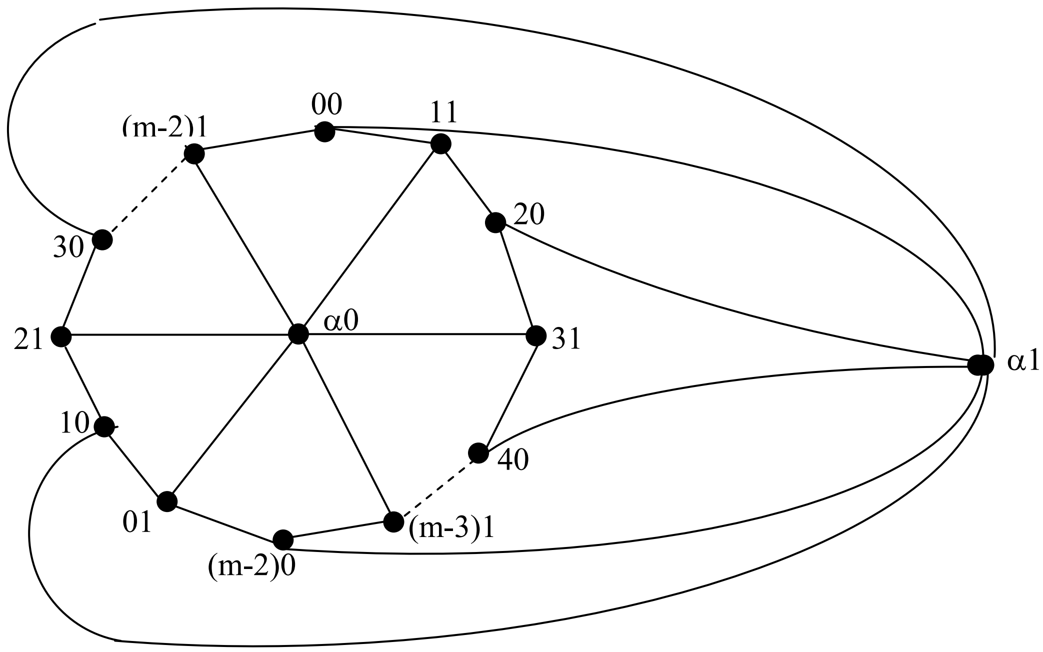
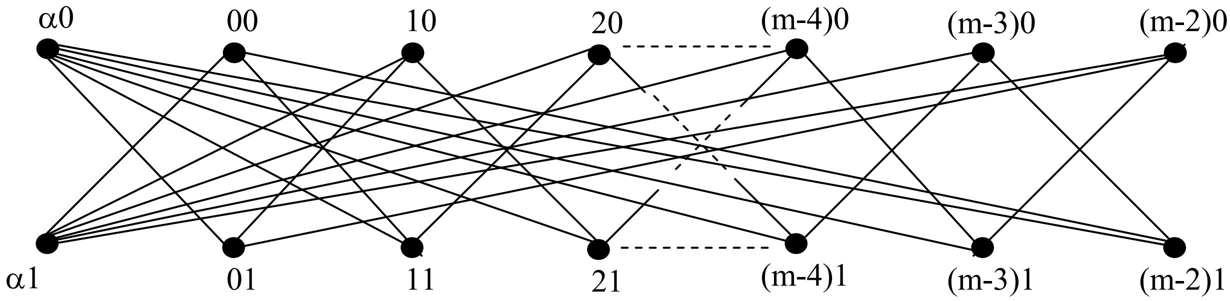


Fig.3

Hence $b(W_m \otimes P_2) = 2$ for even $m \geq 4$.

While, if " *m*" is odd and $m \geq 5$, then $W_m \otimes P_2$ contains a subgraph *K* homeomorphic to $K_{3,3}$. Therefore by MacLanes theorem [5],the graph $W_m \otimes P_2$ is nonplanar (see Fig.4).



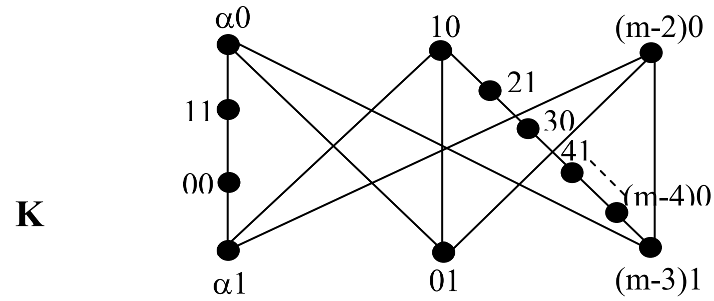


Fig.4: $W_m \otimes P_2$

Hence $b(W_m \otimes P_2) = 3$ for even $m \geq 5$.

We conclude the following table

m	n	$b(W_m \otimes P_n)$
$m \geq 4$, m is even	2	2
$m \geq 4$, m is even	$n \geq 3$	3 or 4
$m \geq 5$, m is odd	2	3
$m \geq 5$, m is odd	$n \geq 3$	3 or 4

2.2. On the Basis Number Of $W_m \otimes C_n$.

In this section, we obtain upper and lower bounds for the basis number of kronecker product of a wheel with a cycle.

Theorem 3. For $m \geq 4$, $n \geq 3$, we have $3 \leq b(W_m \otimes C_n) \leq 5$.

Proof: Since $W_m \otimes P_n$ is a subgraph of $W_m \otimes C_n$ for all $m \geq 4$,and $n \geq 3$, then by Theorem 2, we have $W_m \otimes C_n$ is nonplanar and so by MacLanes theorem [5], we have $b(W_m \otimes C_n) \geq 3$.For *m*=4 and *n*=3,(see Fig.5).

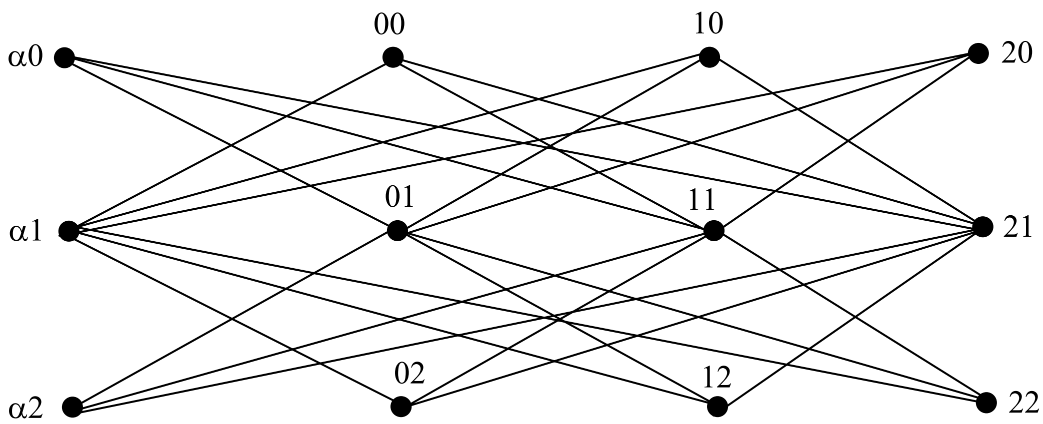


Fig.5: $W_4 \otimes P_3$

To complete the theorem we establish a 5-fold basis $B(W_m \otimes C_n)$ for $\xi(W_m \otimes C_n)$. We have two possibilities for *m*.

(1) " *m* " is even. Then consider the following set of cycles in $W_m \otimes C_n$:

$B(W_m \otimes C_n) = B(C_{m-1} \otimes C_n) \cup N \cup M$.

Where $B(C_{m-1} \otimes C_n)$ is a basis for $\xi(C_{m-1} \otimes C_n)$ discussed in Theorem 2.3.1[4], where " *m-1* " is odd, that is,

$B(C_{m-1} \otimes C_n) = B(C_{m-1} \otimes P_n) \cup B_1 \cup \{S_1, S_2\}$, where

$B(C_{m-1} \otimes P_n) = \{ij, (i-1)(j+1), i(j+2), (i+1)(j+1) : i \in Z_{m-1} \ \ and \ \ j = 0,1,........., n-3\} \cup \{S\}$,

$S = 00,11,20,31,..............., (m-2)0, 01,10,21,30,............., (m-2)1, 00$

$B_1 = \{ij, (i-1)(j+1), i(j+2), (i+1)(j+1), ij \ : \ i = 0,1,.........., m-3 \quad \mathrm{mod}(m-1) \ \ and$
$\ \ j = n-2, n-1 \ \ (\mathrm{mod}\, n)\}$,

$S_1 = (m-2)(n-2), (m-3)(n-1), (m-2)0, 0(n-1), (m-2)(n-2)$

$S_2 = 00, (m-2)(n-1), (m-3)0, (m-4)(n-1),..., 10, 0(n-1), (m-2)0, (m-3)(n-1),..., 20, 1(n-1), 00$.

$N = \{ij, (i+1)(j+1), \alpha j, i(j+1), (i+1)j, \alpha(j+1), ij : i \in Z_{m-1} \ and \quad j \in Z_n\}$.

and

$M=\{\alpha j,i(j+1),(i+1)j,(i+2)(j+1),\alpha j \ and \ \alpha(j+1),ij,(i+1)(j+1),(i+2)j,\alpha(j+1)$
$:i=0,2,4,....,m-4 \ and \ j\in Z_n\}$.

It is clear that

$$\left|B(W_m\otimes C_n)\right|=mn-n+1+(m-1)n+(m-2)n$$
$$=3mn-4n+1=\gamma(W_m\otimes C_n)$$

We will prove that $B(W_m\otimes C_n)$ is independent. It is clear that $N=\bigcup_{j=0}^{n-2}(N_j\cup P_j)\cup\{N_{n-1}\}$ and $M=\bigcup_{j=0}^{n-2}(M_j)\cup\{M_{n-1}\}$, where $N_j\cup P_j\cup M_j$ for $j=0,1,...,n-2$ are as mentioned in the proof of Theorem 2. As in the proof of Theorem 2, $N\cup M$ is independent. Moreover for all $i\in Z_{m-1}$, $N_{n-1}\cup M_{n-1}$ contains the edge $i(n-1),(i+1)0$, which is not contained in $\bigcup_{j=0}^{n-2}(N_j\cup P_j\cup M_j)$.

Thus $N\cup M$ is independent set of cycles. Furthermore $N\cup M$ is independent from the cycles of $B(C_{m-1}\otimes C_n)$ since for all $j\in Z_n$, if C_i is any cycle generated from cycles of $N\cup M$, then C_i contains the edge of the from $\alpha j,i(j+1) \ or \ \alpha(j+1),ij$ for some $i\in Z_{m-1}$ which is not present in any cycle of $B(C_{m-1}\otimes C_n)$.

Thus $B(W_m\otimes C_n)=B(C_{m-1}\otimes C_n)\cup N\cup M$, is independent and so it is a basis. We now consider the fold of $B(W_m\otimes C_n)$. Partition the edge-set of $W_m\otimes C_n$ into $ij,(i+1)(j+1) \ or \ i(j+1),(i+1)j \ and \ \alpha j,i(j+1) \ or \ \alpha(j+1),ij$ for $i\in Z_{m-1}$ and $j\in Z_n$. Therefore if e is any edge in $W_m\otimes C_n$ of the form $ij,(i+1)(j+1) \ or \ i(j+1),(i+1)j$, then

$$f_{B(C_{m-1}\otimes C_n)}(e)\le 3,\quad f_N(e)\le 1,\quad f_M(e)\le 1$$

and so

$$f_{B(W_m\otimes C_n)}(e)\le 5.$$

While if e is any edge of the form $\alpha j,i(j+1) \ or \ \alpha(j+1),ij$ then

$$f_{B(C_{m-1}\otimes C_n)}(e)=0,\quad f_N(e)\le 2,\quad f_M(e)\le 2$$

and so

$$f_{B(W_m\otimes C_n)}(e)\le 4.$$

Thus, the basis $B(W_m\otimes C_n)$ is of fold 5.

(2) "m" is odd, then consider the following set of cycles in $W_m \otimes C_n$:
$B(W_m \otimes C_n) = B^*(C_{m-1} \otimes P_n) \cup \{F_i, F_i' : i \in Z_{m-1}\} \cup N^* \cup M^*$, where $B^*(C_{m-1} \otimes P_n)$ is a basis for $\xi(C_{m-1} \otimes P_n)$ mentioned in [1, Theorem 1, case (2)] and F_i, F_i' are independent cycles [1, Theorem 2, case (1)]. That is,

$B^*(C_{m-1} \otimes P_n) = \{ij, (i+1)(j-1), (i+2)j, (i+1)(j+1), ij :\ i \in Z_{m-1},\ j = 1,2,...,n-2$ $\textit{and}\ \ (j-i)\ \textit{even}\ \} \cup \{ij, (i+1)(j-1), (i+2)j, (i+1)(j+1), ij\ :\ i \in Z_{m-1},$ $j = 1,2,...,n-2\ \ \textit{and}\ \ (j-i)\ \textit{odd}\} \cup \{00, 11, 20, 31, ..., (m-3)0, (m-2)1, 00\}$,

$F_i = \{0i, 1(i-1), 2i, ..., (m-2)(i-1), 0i\}$,
$F_i' = \{0i, 1(i+1), 2i, ..., (m-2)(i+1), 0i\}$,
$N^* = \{ij, (i+1)(j+1), \alpha j, i(j+1), (i+1)j, \alpha(j+1), ij : i = 0,1,...,m-3 \quad \textit{and} \quad j \in Z_n\}$

and
$M^* = \{\alpha j, i(j+1), (i+1)j, (i+2)(j+1), \alpha j \quad \textit{and} \quad \alpha(j+1), ij, (i+1)(j+1), (i+2)j, \alpha(j+1)$ $: i = 0,2,4,.......m-3 \ \bmod(m-1)\ \ \textit{and}\ \ j \in Z_n\}$. It is clear that

$$\begin{aligned}|B(W_m \otimes C_n)| &= (m-1)(n-2) + 1 + 2(m-1) + (m-2)n + (m-1)n \\ &= mn - 2m - n + 2 + 1 + 2m - 2 + mn - 2n + mn - n \\ &= 3mn - 4n + 1 \\ &= \gamma(W_m \otimes C_n).\end{aligned}$$

As in possibility (1), we can prove that $B(W_m \otimes C_n)$ is a 5-fold basis for $\xi(W_m \otimes C_n)$.

Remark. In contrast to upper bounds of the basis numbers of $W_m \otimes P_n$ and $W_m \otimes C_n$ given in Theorem 2 and Theorem 3, one can conjecture that the upper bound for the basis number of kronecker product of two wheels W_m and W_n is $b(W_m \otimes W_n) \le 10$.

Conjecture:

(i) What is $b(K_m \otimes K_n)$? where W_m and W_n are subgraphs of K_m and K_n resp.

(ii) What did you conjecture about $b(G_1 \otimes G_2)$?

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