Construction of Lacunary Sextic spline function Interpolation and their Applications

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الخلاصة:

إن هدف هذا البحث هو الحصول على الوجودِ والوحدانيةِ والحدودِ العليا مِن الأخطاءِ لدالة سبلاين من الدرجة السادسة. كذلك تحت شروط وقيود مناسبة، تم إثبات وحدانية ووجودية الحل. كذلك تم دراسة تحليل تقارب الدالة مع اشتقاق قيود الأخطاء حيث تم استخدام دالة Spline لإيجاد قيمة تقريبية للدالة المعطاة مع حساب مشتقاتها من خلال الدرجة السادسة. تبين لنا كفاءة وقابلية التقنية الجديدة من خلال المثال العددي.

Abstract

In this paper, we consider the construction of the sextic splines function which interpolating the lacunary data. Also, under suitable conditions, we show that the existence and uniqueness of the solution. The convergence analysis of this spline function is studied and the error bounds are derived. This spline function applied to find an approximate value of a given function and its derivatives through six orders. A numerical example has been given to show the applicability and efficiency of the new proposed technique.

Keywords: Interpolated polynomials, Spline functions, Convergence analysis.

1. Introduction

Spline functions are well known and are widely used for practical approximation of functions or more commonly for fitting smooth curves through knot points and have the advantage over many approximation and interpolation techniques in that they are computational feasible. There are a great number of techniques developed for various instances of this problem, such as polynomial regression, wavelets, and from the view point of differential geometry, developable surfaces are composed of general cylinders and cones (see e.g. Yang (2006); Bawa (2005); Kahn and Aziz (2003) and Kurt (1991)) studied the algorithm for cardinal interpolation based on a representation of the Fourier transform of the fundamental interpolation. Also Loghmani and Alavizadeh (2007); Siddiqi, etal (2008)) studied the boundary value problems by various methods.

Fawzy(1987); Venturino(1996), Jwamer(2001) and Al-Bayati etal (2008) presented several local methods for solving lacunary interpolation problems using the different cases. In this study, we derive an algorithm to solve a special lacunary interpolation problem by using sextic spline function, when the function values and its second and fourth derivatives are known at a set of nodes, and also we show that this type of construction of spline functions which interpolates the lacunary data is useful in approximating complicate function and their derivatives on the given interval. The applicability of this spline functions in practical applications checked by one numerical example.

This paper is organized as follows: In section 2, we construct by a similar manner of Fawzy (1987) a sextic spline function which interpolates the lacunary data (0, 2, 4), some theorems about existence and uniqueness of a sextic spline function are also studied. In section 3, some theorems about error bounds and convergence analysis of the new technique are proved. To demonstrate the convergence of the prescribed lacunary spline functions some numerical results are presented in section 4. Last section, deals with the conclusions.

2. The approximation of the spline functions

Descriptions of the method: Let (S_m, C^k) be the class of spline functions with respect to the set of knots x_i. This class consists of piecewise polynomial functions of degree m, which are smoothly connected in the knots, up to the order $k(k<m)$. The spline functions will denoted by $S_i(x)$, where $i=0,1,\ldots,m$.

Now we are concerned with the spline interpolation problem:

Theorem 2.1 Given the real numbers $\{f_k, f_k'', f_k^{(4)}\}_{k=0}^{n}$ such that

$$S(x_k)=f_k,\ S''(x_k)=f_k'' \text{ and } S^{(4)}(x_k)=f_k^{(4)}, \tag{1}$$

where $k=0,1,\ldots,n-1$.

Then the spline approximation has the form

$$S_k(x)=\sum_{j=0}^{6}\frac{(x-x_k)^j}{j!}S_k^j\,, \qquad (2)$$

where $x_k \le x \le x_{k+1}$ and $k=0,1,\ldots,n-1$

Proof: We shall define each of the spline functions $S_k^{(j)}(x)$ explicitly in terms of the data. We choose

$$S_k(x_k)=f_k=f(x_k),$$

$$S_k''(x_k)=f_k''=f''(x_k),$$

and

$$S_k^{(4)}(x_k)=f_k^{(4)}=f^{(4)}(x_k),$$

where $k=0,1,\ldots,n-1$.

From Taylor's series expansion of $f(x)$ and after some derivation with $S(x)$, we obtain the following:

$$S_k^{(6)}=\frac{2}{h^4}\left[f''_{k+2}-3f''_{k+1}+3f''_k-f''_{k-1}\right]+\frac{1}{h^2}\left[f_{k-1}^{(4)}-f_{k+1}^{(4)}\right], \qquad (3)$$

$$S_k^{(5)}=\frac{2}{h^3}\left[f''_{k+2}-2f''_{k+1}+f''_k\right]-\frac{1}{h}\left[f_{k+1}^{(4)}+f_k^{(4)}\right]-\frac{2h}{3}S_k^{(6)}, \qquad (4)$$

$$S_k^{(3)}=\frac{-1}{3h^3}\left[f''_{k+2}-5f''_{k+1}+4f''_k\right]+\frac{h}{6}\left[f_{k+1}^{(4)}-2f_k^{(4)}\right]+\frac{5h^3}{72}S_k^{(6)}, \qquad (5)$$

and

$$S_k'=\frac{1}{h}\left[f_{k+1}-f_k\right]+\frac{h}{180}\left[7f''_{k+2}-44f''_{k+1}-53f''_k\right]+$$
$$\frac{h^3}{360}\left[8f_k^{(4)}-7f_{k+1}^{(4)}\right]-\frac{h^5}{135}S_k^{(6)}. \qquad (6)$$

Also for the first interval $[x_0, x_1]$, we take:

$$S_0^{(6)}=S_1^{(6)}, \qquad (7)$$

$$S_0^{(5)}=S_1^{(5)}-hS_1^{(6)}, \qquad (8)$$

$$S_0^{(3)}=S_1^{(3)}-hS_1^{(4)}+\frac{h^2}{2}S_1^{(5)}-\frac{h^3}{6}S_1^{(6)}, \qquad (9)$$

and

$$S_0'=\frac{1}{h}\Big[f_1-f_0-\frac{h^2}{2}f_0''-\frac{h^4}{24}f_0^{(4)}+\frac{h^4}{6}f_1^{(4)}-\frac{h^3}{6}S_1^{(3)}-$$
$$\frac{11h^5}{120}S_1^{(5)}+\frac{5h^6}{144}S_1^{(6)}\Big]. \qquad (10)$$

Finally for $k=n-1$, we have

$$S_{n-1}^{(j)}=S_{n-2}^{(j)}(x_{n-1}),\ \text{where } j=1,3,5,6. \qquad (11)$$

Now the spline function $S(x)$ which is defined in equations (1)-(2) solves the (0,2,4) lacunary data. Also the construction showed that S is a piecewise sextic polynomial.

Indeed, $S(x)$ is the unique piecewise sextic polynomial in $C^{0,2,4}[x_0,x_1]\cap C^6[x_{n-2},x_n]$

satisfying the interpolation condition (1), we refer to it as lacunary g-spline, See (Fawzy, 1987).
The proof of theorem 2.1 is completed.

3. Consistency relation and convergence analysis:

In this section, the convergence analysis of the new technique has been proved. However, it is essential to determine the order of convergence, and also convenient to write the function using Taylor's expansions, we can establish the following lemma:

Lemma 3.1: For $0 \le k \le n-2$ and $j = 1, 3, 5, 6$, we have

$$\left|S_k^{(j)} - f^{(j)}(x_k)\right| \le c_{kj} h^{6-j} W(D^{(6)} f; h),$$

where $k = 0, 1, 2, ..., n-1$, $W(D^{(6)} f; h)$ is the modulus of continuity of $f^{(6)}$**,** and then the constants c_{kj} are given in the following table:

	c_{k1}	c_{k3}	c_{k5}	c_{k6}
$k=0$	$\frac{49}{432}$	$\frac{181}{288}$	$\frac{3}{2}$	$\frac{9}{4}$
$1 \le k \le n-2$	$\frac{117}{4320}$	$\frac{85}{288}$	$\frac{7}{6}$	$\frac{9}{4}$
$k = n-1$	$\frac{17}{6480}$	$\frac{7}{216}$	$\frac{2}{9}$	1

Proof: First for $k=0$ on the interval $[x_0, x_1]$ and from equations (3)-(10), we obtain

$$S_0' = \frac{1}{h}[f_1 - f_0] + \frac{h}{540}[22 f_3'' - 225 f_2'' - 294 f_1'' - f_0''] +$$
$$\frac{h^3}{1080}\left[-22 f_2^{(4)} + 339 f_1^{(4)} - 46 f_0^{(4)}\right],$$

and

$$S_0' - f_0' = \frac{1}{h}\left[f_1 - f_0 - hf_0'\right] + \frac{h}{540}[22 f_3'' - 225 f_2'' - 294 f_1'' - f_0''] +$$
$$\frac{h^3}{1080}\left[-22 f_2^{(4)} + 339 f_1^{(4)} - 46 f_0^{(4)}\right].$$

After using Taylor's expansion for f_3'', f_2'', f_1'', $f_2^{(4)}$ and $f_1^{(4)}$ about x_0 in the above equation, we obtain

$$\left|S_0' - f_0'\right| \le \frac{49}{432} h^5 W(D^{(6)} f; h).$$

Similarly, we can find

$$\left|S_0''' - f_0'''\right| \le \frac{181}{288} h^3 W(D^{(6)} f; h),$$

$$\left|S_0^{(5)} - f_0^{(5)}\right| \le \frac{3}{2} h W(D^{(6)} f; h),$$

and

$$\left|S_0^{(6)} - f_0^{(6)}\right| \le \frac{9}{4} W\,(D^{(6)} f\,; h)\,.$$

By the same technique as above, on the intervals $[x_k, x_{k+1}]$ for $k = 0,1, 2,...,n-1$, and using Taylor's expansion for f''_{k+3}, f''_{k+2}, f''_{k+1}, $f^{(4)}_{k+2}$ *and* $f^{(4)}_{k+1}$ about x_k, we shall obtain all constants c_{k1}, c_{k3}, c_{k5} and c_{k6}.

Lemma 3.2: Let $f(x) \in C^{m+1}[0,b]$ and $S(x)$ be the lacunary g-spline function constructed (2)-(11). Then the following order of convergence is obtained; i.e.

$$\left\|S^{(m-i)}(x_k) - f^{(m-i)}(x_k)\right\| = O(h^{i+1}), \text{ where } x \in [0,b], i = 0,1,...,5 \qquad (12)$$

and

$$\left\|S\ (x) - f(x)\right\| = O(h^6)\,. \qquad (13)$$

Proof: Let $x_k \le x \le x_{k+1}$ and $z = x - x_k$, then Taylor's expansion for $f(x)$ leads to

$$f(x) = \sum_{n=0}^{m-1} \frac{w^n}{n!} f^{(n)}(x_k) + \frac{z^m}{m!} f^{(m)}(\xi), \text{ for } \quad x_k \le \xi \le x_{k+1} \qquad (14)$$

and

$$S(x) = \sum_{n=0}^{m-1} \frac{w^n}{n!} S^{(n)}(x_k) + \frac{z^m}{m!} S^{(m)}(\xi) \text{ for } \quad x_k \le \xi \le x_{k+1}\,. \qquad (15)$$

Subtracting (14) from (15), provided that $S^{(m)}(x)$ is a constant over $x_k \le x \le x_{k+1}$, and it is zero where $m = 0, 2$ and 4, we obtain

$$\left\|S\ (x) - f(x)\right\| \le \sum_{n=0}^{m-1} \frac{z^n}{n!} \left\|S^{(n)}(x) - f^{(n)}(x)\right\| + \frac{z^m}{m!} \left\|S^{(m)}(\xi) - f^{(m)}(\xi)\right\|.$$

If we make use of the subinterval, given that $|x_k - x| \le h$, $|x_{k+1} - x| \le 2h$ and $|x_{k+3} - x| \le 3h$, then

$$\left\|S\ (x) - f(x)\right\| \le \sum_{n=0}^{m-4} \frac{z^{2n+1}}{(2n+1)!} \left\|S^{(2n+1)}(x) - f^{(2n+1)}(x)\right\| + \frac{z^m}{m!} \left\|S^{(m)}(\xi) - f^{(m)}(\xi)\right\|,$$

for $x_k \le \xi \le x_{k+1}$.

Equation (13) has been proved. To prove equation (12), we define the function $S^{(m-i)}(x)$ which are piecewise continuous on $[0,b]$, and using Taylor's expansion of $f(x)$ of order $(m-i)$, shall obtain

$$S^{(m-i)}(x) = y^{(m-i)}(x) + O(h^{i+1}), \text{ where } i = 0,1,...,5\,.$$

Then

$$\left\|S^{(m-i)}(x) - y^{(m-i)}(x)\right\| = O(h^{i+1}), \text{ where } i = 0,1,...,5\,.$$

This completes the proof of lemma 3.2.

Note: Similar results to Lemma 3.2, was proved under different conditions by Sallam and EL-Hawary (1984) and Fawzy (1987).

Theorem 3.1: Let $f(x) \in C^6[x_0, x_n]$ and let $S(x)$ be the lacunary g-spline function constructed by the equation (2)-(11). Then for all $j \in [0,6]$ and all $k \in [0, n-1]$

$$\left\| S^{(j)}(x_k) - f^{(j)}(x_k) \right\|_{L_\infty[x_k, x_{k+1}]} \le b_{kj} h^{6-j} W(D^{(6)} f; h),$$

where the constants b_{kj} are given in the following table:

	b_{k0}	b_{k1}	b_{k2}	b_{k3}	b_{k4}	b_{k5}	b_{k6}
$k=0$	$\frac{101}{4320}$	$\frac{4397}{8640}$	$\frac{35}{36}$	$\frac{505}{288}$	$\frac{21}{8}$	$\frac{15}{4}$	$\frac{9}{4}$
$1 \le k \le n-2$	$\frac{77}{864}$	$\frac{679}{2880}$	$\frac{7}{12}$	$\frac{361}{288}$	$\frac{55}{24}$	$\frac{41}{12}$	$\frac{9}{4}$
$k = n-1$	$\frac{77}{3240}$	$\frac{59}{1620}$	$\frac{1}{9}$	$\frac{67}{216}$	$\frac{13}{18}$	$\frac{11}{9}$	1

Proof: Suppose that $1 \le k \le n-2$ and let $x_k \le x \le x_{k+1}$. Then using Taylor's expansion of $f(x)$ and the spline function in equation (2.2), we have

$$\left| S(x) - f(x) \right| = \left| S_k(x) - f(x) \right| \le \sum_{n=0}^{5} \frac{h^n}{n!} \left| S^{(n)}(x) - f^{(n)}(x) \right| + \frac{h^6}{6!} \left\| S_k^{(6)} - f^{(6)}(\xi_k) \right\|$$

$$\le \sum_{n=0}^{5} \frac{h^n}{n!} \left| S^{(n)}(x) - f^{(n)}(x) \right| + \frac{h^6}{6!} \left\| S^{(6)}(\xi_k) - f^{(6)}(\xi_k) \right\|, \qquad (16)$$

where $x_k \le \xi_k \le x_{k+1}$.

Now from equation (16) for $1 \le k \le n-2$, and using Lemma 3.1, we obtain

$$\left| S(x) - f(x) \right| = \left| S_k(x) - f(x) \right| \le \frac{77}{864} W(D^{(6)} f; h),$$

$$\left| S_k'(x) - f'(x) \right| \le \frac{697}{2880} W(D^{(6)} f; h),$$

$$\vdots$$

$$\left| S_k^{(6)}(x) - f^{(6)}(x) \right| \le \frac{9}{4} W(D^{(6)} f; h).$$

will easily compute all other constants for $1 \le k \le n-2$.

For $k = 0$ and $x \in [x_0, x_1]$, using the same technique in Lemma 3.1, from equation (16), we obtain

$$\left| S_0(x) - f_o(x) \right| \le \frac{101}{4320} W(D^{(6)} f; h)$$

$$\vdots$$

$$\left| S_0^{(6)}(x) - f_0^{(6)}(x) \right| \le \frac{9}{4} W(D^{(6)} f; h),$$

where $x_0 \le \xi_k \le x_1$.

Finally for $k = n-1$ and for Lemma 3.1 and equation (16), we obtain

$$\left| S_{n-1}(x) - f(x) \right| \le \frac{77}{3240} W(D^{(6)} f; h),$$

$$\vdots$$

$$\left| S_{n-1}^{(6)}(x) - f^{(6)}(x) \right| \le W(D^{(6)} f; h),$$

where $x_{n-2} \le \xi_k \le x_{n-1}$.

This completes the proof of Theorem 3.1

3.4 Outlines of the new sextic spline Algorithm:

Step 1: Partition $[a,b]$ into N subintervals I_k.

Step 2: Set

$S(x_k) = f_k = f(x_k),\ S''(x_k) = f_k'' = f''(x_k)$ and $S^{(4)}(x_k) = f_k^{(4)} = f^{(4)}(x_k)$,

where $k = 0,1,...,n-1$.

Step 3: For $k = 0,1,...,n-1$ do equations (3)-(6) or apply Lemma 3.1.

Step 4: If $x \in [x_{i-1}, x_i]$ go to step 6, else i=i+1 and repeat this until find a proper i.

Step 5: Set k=i.

*Step*6: Computing all pieces of the spline function at N equally spaced points in each subinterval$[x_i, x_{i+1}]$, i=1, 2,…, N or applied Theorem 3.1 .

Step 7: Stop.

4. Numerical illustrations

To illustrate our new technique and to demonstrate the applicability of our presented work computationally, we compare the spline solution of equations (3)-(6), with the analytic value for the test problem 4.1. The numerical results are depicted in tables 4.1A and 4.1B.

Problem 4.1: Let $f(x) = x(1-x)\exp(x)$ where $0 \le x \le 1$. (Siddiqi etal)

Table 4.1A: absolute errors by using Lemma 3.1.

h	$\|S' - f'\|_\infty$	$\|S^{(3)} - f^{(3)}\|_\infty$	$\|S^{(5)} - f^{(5)}\|_\infty$	$\|S^{(6)} - f^{(6)}\|_\infty$
0.05	3.9×10^{-3}	4.1×10^{-1}	12.6×10^{-1}	21.4×10^{-1}
0.015	3.4×10^{-4}	1.2×10^{-1}	3.6×10^{-1}	6.2×10^{-1}
0.01	1.5×10^{-4}	8.07×10^{-2}	2.4×10^{-1}	4.1×10^{-1}
0.005	3.7×10^{-5}	2×10^{-2}	12×10^{-2}	20.5×10^{-2}
0.001	1.5×10^{-6}	8×10^{-3}	2.4×10^{-2}	4×10^{-2}
0.0001	1.5×10^{-8}	8×10^{-4}	2.4×10^{-4}	4.6×10^{-3}

Table 4.1B: Absolute errors by using theorem 3.1.

h	$\|S - f\|_\infty$	$\|S' - f'\|_\infty$	$\|S'' - f''\|_\infty$	$\|S^{(3)} - f^{(3)}\|_\infty$	$\|S^{(4)} - f^{(4)}\|_\infty$	$\|S^{(5)} - f^{(5)}\|_\infty$	$\|S^{(6)} - f^{(6)}\|_\infty$
0.05	5.4×10^{-2}	8.3×10^{-3}	1.13×10^{-1}	8.4×10^{-1}	8.4×10^{-1}	25.7×10^{-1}	21.4×10^{-1}
0.015	1.5×10^{-2}	6.9×10^{-4}	2.7×10^{-2}	2.4×10^{-1}	2.3×10^{-1}	7.3×10^{-1}	6.2×10^{-1}
0.01	1×10^{-2}	3×10^{-4}	1.7×10^{-2}	1.6×10^{-1}	1.5×10^{-1}	4.8×10^{-1}	4.1×10^{-1}
0.005	5×10^{-3}	7.5×10^{-5}	8.6×10^{-3}	8.03×10^{-2}	7.5×10^{-2}	24.1×10^{-2}	20.5×10^{-2}
0.001	1×10^{-3}	3×10^{-6}	1.6×10^{-3}	16×10^{-3}	1.5×10^{-2}	4.8×10^{-2}	4×10^{-2}
0.0001	1×10^{-4}	3×10^{-8}	1.6×10^{-4}	16×10^{-4}	15×10^{-4}	48×10^{-4}	46×10^{-4}

5. Conclusion:

In this paper, we have studied the existence and uniqueness of the sextic spline function that matches function values, second and fourth derivatives at the knots. Also, the error estimate was derived theoretically and examined numerically to show that our construction of the spline function for interpolating lacunary data was efficient.

References

1) Al-Bayati, A.Y., Saeed, R. K. and Jwamer, K. H., The Existence, Uniqueness and upper bounds for errors of six degree spline interpolating the lacunary that (0, 2, 5). Accepted for publication in the J. Comput. Sci. and Math. Mosul Iraq, 2008.

2) Bawa, R. K., Spline based computational technique for linear singularity perturbed boundary value problems, App. Math. Comput. 167,225-236, 2005.

3) Eamail, M. N.; Fawzy, Th.; Ahmed, M. and Elmoselhi, H. O., Deficient spline function approximation to fourth order differential equations, Appl. Math. Modeling, 18, 658-664, 1994.

4) Fawzy, Th., Notes on lacunary interpolation with splines III, Acta Math. Hung. 50, 33-37, 1987.

5) Jetter, K. and Stockler, J., Algorithm for cardinal interpolation using box splines and radial functions, Numer. Math., 60, 97-114, 1991.

6) Jwamer, K. H-F., Study some of the lacunary interpolation by splines and their error bounds, M. Sc. Thesis, Salahaddin University/Erbil, Iraq, 2001.

7) Khan, A. and Aziz, T., Parametric cubic spline approach to the solution of a system of second order boundary value problems. J. Optim. Theory Appl. 118,45-54, 2003.

8) Loghmani, G. B. and Alavizadeh, S. R., Numerical solution of fourth order problems with separated boundary conditions, Appl. Math. and Comput., 191, 571-581, 2007.

9) Micula, G. Approximate solution of differential equation $y^{(2)} = f(x,y)$ with spline functions, Math. Comput., 27, 807-816, 1973.

10) Sallam, S. and EL-Hawary, H. M., A deficient spline function approximation of second order differential equations, Appl. Math. Modeling. 8, 408-412, 1984.

11) Siddiqi, S. S.; Akram, G. and Elahi, A., Quartic spline solution of linear fifth order boundary value problems, Appl. Math. and Comput. 196, 214-220, 2008.

12) Venturino, E., On the (0, 4) lacunary interpolation problem and some related questions, J. of Comput. and Appl. Math. 76, 287-300, 1996.

13) Yang, X. and Yang, W., Cone spline approximation via fat conic spline fitting, Computer-Aieded Design 38, 703-712, 2006.