

On Equivalence Relation in Factor Algebras

Yassin A. Salih

Department of Operations Research
College of Computers and Mathematics Seines
University Mosul

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الخلاصة

في هذا البحث تم إدخال ودراسة العلاقة "≈" بين المساقط، العلاقة "≈" على المؤثرات الموجبة والعلاقة "≈_G" (G- زمرة) بين المساقط في جبر العوامل الموسعة. النتائج الأساسية في هذا البحث تنص على أن كل من هذه العلاقات هي علاقة تكافؤ (المبرهنة 2.1) و (المبرهنة 2.2). أخيراً حددنا العلاقة بين هذه المفاهيم.

Abstract

In this paper we introduce and study a relation "≈" between projections, a relation "≈" on positive operators and a relation "≈_G" (G-group) on projections in extended factor algebras. The main results in this paper state that these relations are indeed an equivalence relation (Theorem 2.1) and (Theorem 2.2). Finally, we determine the relation between these concepts.

1. Introduction

In this work three kinds of relations are introduced and studied. The first one [Def.1.4] between projections, the second relation [Def.1.5] applicable to all positive operators and the third relation [Def.1.6] between projections on extended factor algebra.

We refer the reader to the references [1], [2], [4], [6] and [7], as standard background references for C^* - and von Neumann algebras and for the subject.

Following [2],[6] we introduce some concepts which will be use in various places in this paper. We denote by $B(H)$ as the algebra of all bounded linear operators acting on a Hilbert space H and let A be a subset of $B(H)$. By A' we denote the set of all elements of $B(H)$ commuting with

every element in A , then A' is called the commutant of A . Let $(A')' = A''$, A'' is called the bicommutant of A . It is clear that A' is Banach algebra containing the identity I . If $x \in A$ implies $x^* \in A$, then A' is C^* -algebra acting on H , where $*$ is involution (or $*$ -operator).

Recall that an algebra A is said to be semiprime if for any ideal I of A such that $I^2 = 0$ implies that $I = 0$.

1.1. Definition: [6]

$*$ -subalgebra A of an algebra $B(H)$ is called a von Neumann algebra on H if $A = A''$. For example $B(H)$ is von Neumann algebra.

1.2. Definition:

Let A be a semiprime von Neumann algebra, the extended centroid of A is denoted by $C(A)$ and defined by

$$C(A) = \{f : I \rightarrow A \mid f(xa) = f(x)a, f(ax) = af(x), \forall a \in A, x \in I\},$$

where I is a nonzero ideal of A and f is a linear mapping.

Let A be a semiprime von Neumann algebra with identity, we recall that an element $a \in A$ is self-adjoint or hermitian if $a = a^*$, unitary if $a^*a = aa^* = I$ when A contains the identity I , a projection if $a^2 = a$ and $a^* = a$, by A^+ we denoted the positive portion of A , i.e. $A^+ = \{a^2 : a \in A\}$.

We call the extended centroid of semiprime von Neumann algebra by an extended factor algebra.

1.3. Definition: [6]

Let e and f be two projections in extended factor algebra F , e is said to be equivalent to f (written $e \sim f$), if there exists a partial isometry v in F such that $e = v^*v$ and $f = vv^*$.

Observe that if v^*v is a projection, then vv^* is automatically also a projection.

If $e \sim e_1 \leq f$ for some projection $e_1 \in F$, then we write $e \lesssim f$.

1.4. Definition: [3]

Let F be an extended factor algebra and let s and t in F^+ , we write $s \approx t$ if there is a set of elements $\{a_i\}$ in F such that

$$s = \sum a_i^* a_i \quad \text{and} \quad t = \sum a_i a_i^*$$

and we say s is equivalent to t .

In other words, $s \approx t$ if s and t are the l.u.b. for the sets of finite sums of $a_i^* a_i$'s and $a_i a_i^*$'s respectively.

For s and t in F^+ , we write $s \lesssim t$ if there is r in F^+ such that $s \approx r \leq t$.

Now let A be a semiprime von Neumann algebra acting on a Hilbert space H . Let G be a group and $t \rightarrow u_t$ a unitary representation of G on H such that $u_t^* A u_t = A$ for all $t \in G$.

1.5. Definition: (G -equivalence) [5]

Let F be an extended factor algebra. If e and f are projections in F we say, e is G -equivalent to f (written $e \sim_G f$) if for each $t \in G$ there is an operator $k_t \in F$ such that

$$e = \sum_{t \in G} k_t k_t^* \quad \text{and} \quad f = \sum_{t \in G} u_t^* k_t^* k_t u_t.$$

For two projections e and f in F we write $e \underset{G}{\lesssim} f$ if $e \sim_G f_o \leq f$.

2. The main results:

In this section we give the main results which state that the relations " \sim ", " \approx " and " \sim_G " are indeed an equivalence as shown in the following theorems. One can easily see that the relation " \sim " satisfies the conditions of equivalence.

2.1 Theorem:

Let F be an extended factor algebra. The relation " \approx " is an equivalence relation in F^+ .

Proof:

- (i). Clearly $s \approx s$ for any s in F^+ .
- (ii). Let $s \approx t$, then there exist $\{a_i\} \subset F$ such that

$$s = \sum a_i^* a_i \quad \text{and} \quad t = \sum a_i a_i^*.$$

Put $b_i = a_i^*$, then we get

$$s = \sum b_i b_i^* \quad \text{and} \quad t = \sum b_i^* b_i.$$

Hence we get $t \approx s$.

- (iii). To prove the transitivity, let $s \approx t$ and $t \approx r$, then there exists two sets $\{a_i\}$ and $\{b_j\}$ in F such that

$$s = \sum a_i^* a_i, \quad t = \sum a_i a_i^* = \sum b_j^* b_j, \quad r = \sum b_j b_j^*.$$

Using [3, Prop. 2.4], then there is a set $\{c_{ij}\}$ in F such that

$$a_i^* a_i = \sum_j c_{ij}^* c_{ij} \quad \text{and} \quad b_j b_j^* = \sum_i c_{ij} c_{ij}^*.$$

Now
$$s = \sum_i a_i^* a_i = \sum_i \sum_j c_{ij}^* c_{ij}$$

$$r = \sum_j b_j b_j^* = \sum_j \sum_i c_{ij} c_{ij}^*.$$

Hence we get $s \approx r$ which complete the proof. Δ

In order to prove that " \sim_G " is an equivalence relation on the projections in the extended factor algebra M , we need the following structure. Let A be a simeprime von Neumann algebra acting on a Hilbert

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space H , G is a group and $t \rightarrow u_t$ is a unitary representation of G on H such that $u_t^* A u_t = A$ for all $t \in G$. For $t \in G$ let H_t be a Hilbert space of the same dimension as H and J_t an isometry of H onto H_t . Let $\tilde{H} = \sum_{t \in G} \oplus H_t$. We write an operator $r \in B(\tilde{H})$ (the bounded operators on \tilde{H}) – as a matrix $(r_{s,t})_{s,t \in G}$, where $r_{s,t} = J_s^* r J_t \in B(H)$.

For each $k \in A$ let $\Phi(k)$ denoted the element in $B(\tilde{H})$ with matrix $(r_{s,t})$ where $r_{s,t} = 0$ if $s \neq t$ and $r_{s,s} = k$ for all $s \in G$. Then Φ is a *- isomorphism of A onto a von Neumann subalgebra \tilde{A} of $B(\tilde{H})$. For $y \in G$ let \tilde{u}_y be the operator in $B(\tilde{H})$ with matrix $(r_{s,t})$, where $r_{s,t} = 0$ if $st^{-1} \neq y$, $r_{yt,t} = u_y$ for all $t \in G$.

Then (see [1, Ch. I §9]) $y \rightarrow \tilde{u}_y$ is a unitary representation of G on \tilde{H} such that $\tilde{u}_y^* \Phi(k) \tilde{u}_y = \Phi(u_y^* k u_y)$, $y \in G$, $k \in A$.

Let B denotes the semiprime von Neumann algebra generated by \tilde{A} and the \tilde{u}_y , $y \in G$, then each operator in B is represented by a matrix $(r_{s,t})$ where $r_{s,t} = k_{st^{-1}} u_{st^{-1}}$, $k_{st^{-1}} \in A$. Now we write $p \sim q$ for two projections p and q in B and we mean that they are equivalent, i.e. there is a partial isometry $v \in B$ such that $vv^* = p$, $v^*v = q$.

The next theorem includes the fact that " \sim_G " is an equivalence relation and shows more, namely that \sim_G - equivalence is the same as equivalence " \sim " in B .

2.2 Theorem:

Let e and f be projections in extended factor algebra M . Then $e \sim_G f$ if and only if $\Phi(e) \sim \Phi(f)$. Hence \sim_G is an equivalence relation on the projections in M .

Proof:

Suppose $e \sim_G f$. Then for each $t \in G$, there is $k_t \in M$ such that

$$e = \sum_{t \in G} k_t k_t^* \quad \text{and} \quad f = \sum_{t \in G} u_t^* k_t^* k_t u_t.$$

Then we get

$$\begin{aligned} \Phi(e) &= \sum_t \Phi(k_t k_t^*) = \sum_t \Phi(k_t) \Phi(k_t)^* \\ &= \sum_t (\Phi(k_t) \tilde{u}_t) (\Phi(k_t) \tilde{u}_t)^* \end{aligned}$$

and

$$\begin{aligned} \Phi(f) &= \sum_t \Phi(u_t^* k_t^* k_t u_t) = \sum_t \tilde{u}_t^* \Phi(k_t^* k_t) \tilde{u}_t \\ &= \sum_t (\Phi(k_t) \tilde{u}_t)^* (\Phi(k_t) \tilde{u}_t). \end{aligned}$$

Thus by [3, Th. 4.1] $\Phi(e) \sim \Phi(f)$.

Conversely assume $\Phi(e) \sim \Phi(f)$. Then there is a partial isometry $v \in M$ such that

$$vv^* = \Phi(e) \quad , \quad v^*v = \Phi(f).$$

Let $v = (k_{s_{t^{-1}}} u_{s_{t^{-1}}})$, then an easy calculation show

$$e = \sum_{t \in G} k_t k_t^* \quad \text{and} \quad f = \sum u_t^* k_t^* k_t u_t,$$

hence $e \sim_G f$ and the proof is complete. Δ

Throughout the following notes we can obtain an interesting and important relation between the equivalences.

Note 1:

If G is the one element group, then " \sim_G " implies to " \sim ".

Note 2:

If G is the additive group and the representation $t \rightarrow u_t$ is the trivial representation, i.e. $u_t = I$ for $t \in G$, then " \sim_G " implies to " \approx ".

Note 3:

If e and f are equivalent projections in extended factor algebra F , i.e. there is a partial isometry $v \in F$ such that $e = vv^*$, $f = v^*v$, then $e \sim_G f (e \sim f \Rightarrow e \sim_G f)$. Clearly from the definition of \sim_G by putting $k_e = v$, $k_t = 0$ for $t \neq e$.

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