A Local Existence Theorem For Certain Fractional Differential Equations

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الملخص

تناولنا في هذه الدراسة وجود الحل للمعادلة التفاضلية اللاخطية ذات الرتبة الكسرية α ومن الصيغة:

$$y^{(\alpha)}(x)=f(x,y) \quad , \quad \alpha\in(0,1) \ , \ \alpha\in R$$

والتي تحقق الشرط الابتدائي

$$y^{(\alpha-1)}(x_0)=y_0$$

وكذلك المعادلة التفاضلية غير الخطية

$$y^{(\alpha)}(x)=f(x,y) \quad , \quad n-1<\alpha<n \ , \ n\geq 2 \ , \ n\in Z^+$$

والتي تحقق الشرط الابتدائي

$$y^{(\alpha-i)}(x_0)=y_{i0} \qquad i=1,2,3,.......$$

وباستخدام طريقة شاودر للنقطة الثابتة.

ABSTRACT

In this work we study the existence solution for certain non-linear differential equations of fractional order α of the form

$$y^{(\alpha)}(x)=f(x,y) \quad , \quad \alpha\in(0,1) \ , \ \alpha\in R$$

With

$$y^{(\alpha-1)}(x_0)=y_0$$

$$y^{(\alpha)}(x)=f(x,y) \quad , \quad n-1<\alpha<n \ , \ n\geq 2 \ , \ n\in Z^+$$

With

$$y^{(\alpha-i)}(x_0)=y_{i0} \quad , \quad i=1,2,3,........$$

By using Schauder fixed point theorem.

INTRODUCTION

The concept of fractional differentiation is used in the solution of ordinary Higgins[7], partial Riesz and integral equations Bassam [2] and to find the inverse Laplace transformation Colan and Kolr [3]. Although other methods of solutions are available , the fractional derivative approach to these problems often suggests methods that are not obvious in a classical formulation. A classical example of a differential equation of non-integer order is the inverse of the Abel integral eguation Bocher , that is consider the solution as the differential equation , where the integral equation becomes the solution of this equation . An example of such an equation

$$y^{(\alpha)}(x)+\lambda y=h(x)$$

Was discussed by Barrett [1] .Bassam [2] proved the local existence and uniqueness theorem of the differential equation

$$y^{(\alpha)}(\alpha)=f(x,y) \quad , \quad 0<\alpha<1$$

By using the Banach contraction principle on the same limes as used by Bielecki for an ordinary differential equation. Our work is to extend some results of [6] to prove the existence solution for non-linear fractional differential equation which has the form

$$y^{(\alpha)}(x)=f(x,y) \quad , \quad \alpha\in(0,1) \ , \quad \alpha\in R \qquad \text{..........(1)}$$

With

$$y^{(\alpha-1)}(x_0)=y_0 \qquad \text{..........(2)}$$

and

$$y^{(\alpha)}(x)=f(x,y) \quad , \quad n-1<\alpha<n \quad , \quad n\geq 2 \quad , \quad n\in Z^{+} \qquad \text{..........(3)}$$

With

$$y^{(\alpha-i)}(x_0)=y_{0i} \quad . \qquad i=1,2,3,....... \qquad \text{..........(4)}$$

Where $x\in J=\{x:r_1<x<r_2:r_1,r_2\in R\}$ and f is a function from $J\times R$ into R , where R is the set of real numbers, and y_0, y_{i0} are constants, and it is continuous on the domain

$$D:\left\{(x,y):0<x_0<x\leq x_0+c, \left\| y-\frac{y_0(x-x_0)^{\alpha-1}}{\Gamma(\alpha)}\right\|\leq d\right\} \qquad \text{..........(5)}$$

and the domain

$$D:\left\{(x,y):0<x_0<x\le x_0+c,\left\|y-\sum_{i=1}^{n-1}\frac{y_{i0}(x-x_0)^{\alpha-i}}{\Gamma(\alpha-i+1)}\right\|\le d\right\} \quad \text{..........(6)}$$

Define the norm of $y\in C(J)$ by :

$$\|y\| = \sup_{x\in J}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1}z(t)dt\right|\right)|y(x)|\right\} \quad \text{..........(7)}$$

We define the set $S(\rho)$ as follows:

$$S(\rho)=\{y:y\in C(J):\|y\|\le\rho\} \quad \text{..........(8)}$$

PRELIMINARIES

In this section we set some definitions and lemmas to be used in this work..

Definition 1:

A sequence $\{f_n\}_{n=1}^{\infty}$ in a normed linear space is called a Cauchy sequence if, given $\varepsilon>0$, there is an $N\in I$ such that for all $n,m\ge N$, we have

$$\|f_n-f_m\|<\varepsilon$$

Definition 2:

A complete normed linear space is called a Banach space .

Definition 3:

Let f be a function which is defined a.e (almost every where) on [a,b].For $\alpha>0$, we define:

$$ {}_a^bI^\alpha f=\frac{1}{\Gamma(\alpha)}\int_a^b f(t)(b-t)^{\alpha-1}dt$$

Provided that this integral (Lebesgue) exists , where Γ is the Gamma function.

Lemma 1:

Let $\alpha,\beta\in R$, $\beta>-1$. If $x>a$, then

$$ {}_a^bI^\alpha\frac{(t-a)^\beta}{\Gamma(\beta+1)}=\begin{cases}\dfrac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} & \alpha+\beta\ne \text{ negative integer}\\ 0 & \alpha+\beta= \text{ negative integer}\end{cases}$$

Theorem 1 (Schauder's fixed-point theorem):

If k is a closed ,bounded and convex subset of a Banach space E , and the mapping $T:k\to k$ is completely continuous , then T has a fixed point in k .

Theorem 2:

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real – valued functions on the set E, then $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent on E if and only if given $\varepsilon > 0$ there exist $N \in Z^+$ such that

$$|f_m(x) - f_n(x)| \in \qquad (m, n \geq N, \quad x \in E)$$

Theorem 3:

If $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous real – valued functions on a metric space X that converge uniformly to f on X. then f is also continuous on X.

Theorem 4:

A sequence $\{f_n\}_{n=1}^{\infty}$ in C[a,b] converges to $f \in C[a,b]$ if and only if it is converge uniformly to f on [a,b].

Remark1:

For the proof of definitions, lemmas and theorems see [1], [2], [4], [5], and [8].

THE MAIN THEOREMS

Theorem 5 (Local Existence theorem):

If f(x,y) is a continuous in a closed ,bounded rectangular box (5) .Then there exist at least one solution y(x) of (1) and (2) on : $0 < x_0 < x \leq x_0 + \beta$, for some $\beta > 0$.

Proof:

Let $C(J) = C[J, R]$ be the space of all continuous functions from : $J = (x_0, x_0 + c]$ in to R.

We shall prove that the space C(J) with the norm (7) is a Banach space.

Let $\{y_n(x)\}_{n=1}^{\infty}$ be a Cauchy sequence in C (J), $\forall \varepsilon > 0$ there is an $m_0 \in I$ such that

$$\|y_n - y_m\| \leq \varepsilon \quad , \quad (m, n \geq m_0)$$

$$\sup_{x \in J}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|y_n(x) - y_m(x)|\right\} \leq \varepsilon \quad .$$

Thus

$$\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|y_n(x) - y_m(x)|\right\} \leq \varepsilon$$

or

$$|y_n(x) - y_m(x)| \leq \frac{\varepsilon}{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)}$$

$$|y_n(x) - y_m(x)| \leq \varepsilon_1$$

where $\varepsilon_1 = \dfrac{\varepsilon}{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)}$ and by theorem (2) the sequence of functions $\{y_n(x)\}_{n=1}^{\infty}$ is convergent uniformly to the limit y(x). Also from the theorem (3) $y \in C(J)$ therefore from the theorem (4) $\{y_n(x)\}_{n=1}^{\infty}$ convergent to the y. This implies that C (J) is a Banach space.

Since f is a continuous on the *D*, there exists a positive number M such that:

$$\|f(x,y)\| \le M \quad \text{for } (x,y) \in D \qquad \text{...........(9)}$$

Choose β and ρ such that $0 < \beta \le c$, $0 < \rho \le d$ and $\dfrac{M}{\Gamma(\alpha+1)}\beta^{\alpha} \le \rho$

It is clear that the set (8) is a closed and bounded subset of the Banach space C (J), $S(\rho)$ is convex, for if $y_1, y_2 \in S$ $\quad 0 < \lambda \le 1$ then

$$\|\lambda y_1 + (1-\lambda)y_2\| = \sup_{x\in J}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|\lambda y_1(x) + (1-\lambda)y_2|\right\}$$

$$= \sup_{x\in J}\left\{\left(\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|\lambda y_1(x)|\right) + \right.$$
$$\left. + \left(\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|(1-\lambda)y_2(x)|\right)\right\}$$

$$\le \lambda \sup_{x\in J}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|y_1(x)|\right\} +$$
$$+ (1-\lambda)\sup_{x\in J}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|y_2(x)|\right\}$$

$$\le \lambda\|y_1\| + (1-\lambda)\|y_2\|$$

$$\le \lambda\rho + (1-\lambda)\rho \quad = \rho$$

$$\|\lambda y_1 + (1-\lambda)y_2\| \le \rho$$

So that $\lambda y_1 + (1-\lambda)y_2 \in S(\rho)$

Therefore $S(\rho)$ is convex.

Now, let T be an operator defined by:

$$(Ty)(x) = \frac{y_0(x-x_0)^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1} f(t,y(t))dt \qquad \text{...........(10)}$$

for $y \in S(\rho)$, $x \in (x_0, x_0 + \beta]$.

Since f is continuous , it follows that T is continuous operator from $S(\rho)$ to C(J).

From (9) and (10) , we have:

$$\left\| (Ty)(x) - \frac{y_0 (x - x_0)^{\alpha-1}}{\Gamma(\alpha)} \right\| = \frac{1}{\Gamma(\alpha)} \left\| \int_{x_0}^{x} (x-t)^{\alpha-1} f(t, y(t)) dt \right\|$$

$$\le \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} \| f(t, y(t)) \| dt$$

$$\le \frac{M}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} dt$$

$$\le \frac{M}{\Gamma(\alpha+1)} \beta^{\alpha} \le \rho$$

This implies that *T* maps from $S(\rho)$ into itself.

Now, we show that all functions belonging to $T(S(\rho))$ are uniformly bounded on $(x_0, x_0 + \beta]$.

$$\| (Ty)(x) \| = \left\| \frac{y_0 (x - x_0)^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} f(t, y(t)) dt \right\|$$

$$\le \frac{\|y_0\| \beta^{\alpha-1}}{\Gamma(\alpha)} + \frac{M}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} dt$$

$$= \frac{\|y_0\| \beta^{\alpha-1}}{\Gamma(\alpha)} + \frac{M\beta^{\alpha}}{\Gamma(\alpha+1)}$$

$$\le \frac{\|y_0\| \beta^{\alpha-1}}{\Gamma(\alpha)} + \rho$$

This implies that $T(S(\rho))$ is uniformly bounded.

Next, we shall prove that the functions of $T(S(\rho))$ are equicontinuous on $(x_0, x_0 + \beta]$

To do so, it is enough to show that the integral part of (10) which we shall

$$G(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} f(t, y(t)) dt$$

Forms a family of equicontinuous function on $(x_0, x_0 + \beta]$;because $\frac{y_0 (x - x_0)^{\alpha-1}}{\Gamma(\alpha)}$ is equicontinuous function, since its derivative is uniformly bounded on $(x_0, x_0 + \beta]$

Now, for $x_1, x_2 \in (x_0, x_0 + \beta]$ such that: $0 < x_0 < x_1 < x_2 \le x_0 + \beta$, from (10) we obtain :

$$\|G(x_2) - G(x_1)\| = \frac{1}{\Gamma(\alpha)} \left\| \int_{x_0}^{x_2} (x_2 - t)^{\alpha-1} f(t, y(t))dt - \int_{x_0}^{x_1} (x_1 - t)^{\alpha-1} f(t, y(t))dt \right\|$$

$$= \frac{1}{\Gamma(\alpha)} \left\| \int_{x_0}^{x_1} (x_2 - t)^{\alpha-1} f(t, y(t))dt + \int_{x_1}^{x_2} (x_2 - t)^{\alpha-1} f(t, y(t))dt - \int_{x_0}^{x_1} (x_1 - t)^{\alpha-1} f(t, y(t))dt \right\|$$

$$\le \frac{1}{\Gamma(\alpha)} \left\{ \int_{x_0}^{x_1} \left[(x_2 - t)^{\alpha-1} - (x_1 - t)^{\alpha-1} \right] \|f(t, y(t))\| dt + \int_{x_1}^{x_2} (x_2 - t)^{\alpha-1} \|f(t, y(t))\| dt \right\}$$

$$\le \frac{M}{\Gamma(\alpha)} \left\{ \int_{x_0}^{x_1} \left[(x_2 - t)^{\alpha-1} - (x_1 - t)^{\alpha-1} \right] dt + \int_{x_1}^{x_2} (x_2 - t)^{\alpha-1} dt \right\}$$

(Since $\left[(x_2 - t)^{\alpha-1} - (x_1 - t)^{\alpha-1}\right]$ is negative for all $\alpha \in R$)

By usual integration, we obtain

$$\|G(x_2) - G(x_1)\| \le \frac{M}{\alpha\Gamma(\alpha)} \left\{ 2(x_2 - x_1)^{\alpha} + \left| (x_2 - x_0)^{\alpha} - (x_1 - x_0)^{\alpha} \right| \right\}$$

$$\le \frac{M}{\Gamma(\alpha + 1)} \left\{ 2(x_2 - x_1)^{\alpha} + |x_2 - x_1|^{\alpha} \right\} \qquad \text{..........(11)}$$

From (11) and since $x^{\alpha} \to x_0^{\alpha}$ as $x \to x_0$, i.e. x^{α} is uniformly convergent on $(x_0, x_0 + \beta]$, we obtain that the function of $T(S(\rho))$ are equicontinuous on $(x_0, x_0 + \beta]$. Hence the closure of $T(S(\rho))$ is compact.

Now, from Schauder's fixed-point theorem T possesses a fixed-point.

$$(Ty)(x) = y(x)$$

$$y(x) = \frac{y_0 (x - x_0)^{\alpha-1}}{\Gamma(\alpha)} + \int_{x_0}^{x} (x - t)^{\alpha-1} f(t, y(t))dt$$

$y(x)$ is a solution of the equation (1) and satisfies the initial condition (2).

Theorem 6:
If f(x,y) is a continuous in a closed ,bounded rectangular box (6).Then there exist at least one solution y(x) of (3) and (4) on : $0 < x_0 < x \le x_0 + \beta$, for some $\beta > 0$.

Proof:

Let T be an operator define by:

$$(Ty)(x)=\sum_{i=1}^{n-1}\frac{y_{i0}(x-x_0)^{\alpha-i}}{\Gamma(\alpha-i+1)}+\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}f(t,y(t))dt \qquad \text{..........(12)}$$

for $y\in S(\rho), x\in(x_0,x_0+\beta]$. It is clear that $S(\rho)$ is a closed and bounded subset of the Banach space C (J) and it is convex.

Since f is continuous, it follows that T is continuous operator from $S(\rho)$ to C(J),from (9) and (12) , we have:

$$\left\|(Ty)(x)-\sum_{i=1}^{n-1}\frac{y_{i0}(x-x_0)^{\alpha-i}}{\Gamma(\alpha-i+1)}\right\|=\frac{1}{\Gamma(\alpha)}\left\|\int_{x_0}^{x}(x-t)^{\alpha-1}f(t,y(t))dt\right\|$$

$$\le\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}\|f(t,y(t))\|dt$$

$$\le\frac{M}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}dt$$

$$\le\frac{M}{\Gamma(\alpha+1)}\beta^{\alpha}\le\rho$$

This implies that T maps from $S(\rho)$ into itself.

Now, we show that every functions belonging to $T(S(\rho))$ is uniformly bounded on $(x_0,x_0+\beta]$.

$$\|(Ty)(x)\|=\left\|\sum_{i=1}^{n-1}\frac{y_{i0}(x-x_0)^{\alpha-1}}{\Gamma(\alpha-i+1)}+\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}f(t,y(t))dt\right\|$$

$$\le\sum_{i=1}^{n-1}\frac{\|y_{i0}\|\beta^{\alpha-i}}{\Gamma(\alpha-i+1)}+\frac{M}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}dt$$

$$\le\sum_{i=1}^{n-1}\frac{\|y_{i0}\|\beta^{\alpha-i}}{\Gamma(\alpha-i+1)}+\frac{M}{\Gamma(\alpha+1)}\beta^{\alpha}$$

$$\le\sum_{i=1}^{n-1}\frac{\|y_{i0}\|\beta^{\alpha-i}}{\Gamma(\alpha-i+1)}+\rho$$

This implies that $Ty \in T(S(\rho))$ is uniformly bounded.
In the theorem (5) we prove all functions in $T(S(\rho))$ are equicontinuous on $(x_0, x_0 + \beta]$.Hence the closure of $T(S(\rho))$ is compact.
Now, from Schauder's fixed-point theorem *T* possesses a fixed-point.

$$(Ty)(x) = y(x)$$

$$y(x) = \sum_{i=1}^{n-1} \frac{y_{i0}(x - x_0)^{\alpha - i}}{\Gamma(\alpha - i + 1)} + \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x - t)^{\alpha - 1} f(t, y(t)) dt$$

$y(x)$ is a solution of the equation (3) and satisfies the initial condition (4).

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