On Strongly γ – Regular Rings

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المخلص

الهدف الرئيسي في هذا البحث هو دراسة الحلقات المنتظمة القوية من النمط – γ. والتي ادخل تعريفها (*Mohammad A. J. and Salih. S. M.*) في ٢٠٠٦. الحلقة R تسمى حلقة منتظمة قوية من النمط – γ إذا كان لكل a في R يوجد b في R وعدد صحيح موجب $1 \neq n$ بحيث ان $a = a^2 b^n$.

كذلك درسنا بعض الصفات الرئيسية لهذه الحلقات . أخيراً وضحنا العلاقة بين الحلقات المنتظمة القوية من النمط – γ وبعض الحلقات الأخرى.

ABSTRACT

The main goal of the work is to study a strongly γ-regular rings, which was introduce by Mohammad A. J. and Salih. S. M. in (2006). That is, a ring R is said to be strongly γ-regular if for every $a \in R$ there exists $b \in R$ and a positive integer $n \neq 1$ such that $a = a^2 b^n$.

We will study some basic properties of those rings. Finally, we show the relation between strongly γ-regular rings and other rings.

1. INTRODUCTION

Throughout this paper *R* denotes an associative ring with identity and all modules are unitary right R-modules. Recall that; (1) A right *R*-module *M* is called right principally injective (briefly right P-injective) if for any principal right ideal (*aR*) of *R*, and every right *R*-homomorphism of *aR* into *M* extends to one of *R* into *M*. This concept was introduced by [6,1]; (2) A ring *R* is said to be regular if for every $a \in R$, there exists $b \in R$ such that $a=aba$; (3) A ring *R* is called strongly regular if for every $a \in R$, there exists $b \in R$ such that $a=a^2b$; (4) A ring *R* is called strongly π-regular, if for every $a \in R$ there exists $n \in z^+$ and element $b \in R$ such that $a^n=a^{n+1}\, b(a^n=ba^{n+1})$; (5) For any element *a* in *R* we define the right annihilator of *a* by $r(a)=\{x \in R : ax=0\}$. And likewise the left annihilator *l* (*a*). In 2006 Mohammad A. [4] defined γ-regular rings, that is, *a* ring *R* with every $a \in R$, there exists *b* in *R* and a positive integer $n \neq 1$ such that $a=ab^na$. Also, the definition of strongly γ–regular ring was introduced in [4].

2. Strongly γ-Regular Rings

In this section, we study some basic properties of strongly γ–regular rings.

Definition 2.1 : [4]

An element *a* of a ring *R* is said to be strongly γ-regular if there exists *b* in *R* and a positive integer $n \neq 1$ such that $a=a^2b^n$.

A ring *R* is said to be strongly γ-regular if every element in *R* is strongly γ-regular element.

Hence, in a strongly γ-regular ring *R*, $a=a^2b^n$ if and only if $a = b^na^2$, see [3].

Remark 2.2 : [4]

We see that every strongly γ-regular ring is strongly regular ring, however the converse is not true in general, for example, the ring (Q,+,.) of rational numbers, the rational (real) Hamilton Quaternion and a quadric field are strongly regulars, but not strongly γ-regulars.

Proposition 2.3 :

If *R* is a reduced ring such that for each non zero element $a \in R$ there is a unique $b \in R$ such that $a^n=a^{2n}b$, a positive integer $n \neq 1$, then *b* is strongly γ-regular element.

Proof: Since $a^n=a^{2n}b$ for each $a \in R$, thenwe shall prove that *R* has no divisor of zero. Let $a.\, b=0$, Then $a^n=a^{2n}b=a^{2n-1}\, a.b=a^{2n-1}.0=0$. So $a=0$ (*R* is a reduced ring). *Then* cancellation low holds. $1=a^nb \Rightarrow b=a^nb^2$. Therefore *b* is strongly γ-regular element by [3]. □

Lemma 2.4: [5]

If R is a reduced ring, and if a is a non-zero element in R. then $r(a)=r(a^2)$, and $l(a)=r(a)$.

Theorem 2.5 :

Let R be a reduced ring. If $R/r(a)$ is strongly γ-regular ring for all $a\in R$, then R is strongly γ-regular and γ–regular ring.

Proof: Suppose that $R/r(a)$ is strongly γ-regular ring, then for any $a+r(a)\in R/r(a)$, there exists $b+r(a)\in R/r(a)$ and a positive integer $n\neq 1$ such that $a+r(a)=(a+r(a))^2\,(b+r(a))^n$

$$= (a^2+r(a))\,(b^n+r(a))$$

$$= a^2b^n + r(a)$$

Then $a-a^2b^n\in r(a)$. So $a(a-a^2b^n)=0$. that is $a^2(1-ab^n)=0$. Then $(1-ab^n)\in r(a^2)=r(a)$ [Lemma 2.4]. So, $a(1-ab^n)=0$. Hence $a=a^2b^n$. Therefore R is strongly γ-regular ring. Also, since $(1-ab^n)\in l(a)=r(a)$, *then* $(1-ab^n)\,a=0$. So, $a= ab^na$. Therefore, R is γ-regular ring. □

Proposition 2.6 :

If y is an element of a ring R such that $a-a^2\,y^\alpha$ is strongly γ-regular element then a is strongly regular element, where $1\neq\alpha$ is a positive integer.

Proof: Suppose that $a-a^2\,y^\alpha$ is strongly γ-regular element, then there exists an element $b\in R$ and a positive integer $n\neq 1$ such that:

$$a - a^2y^\alpha = (a - a^2y^\alpha)^2\, b^n$$

now $\quad a - a^2y^\alpha = (a - a^2y^\alpha)\,(ab^n - a^2y^\alpha b^n)$

$$= a^2b^n - a^2y^\alpha ab^n - a^2y^\alpha ab^n + a^2y^\alpha a^2y^\alpha b^n$$

Then $\quad a = a^2y^\alpha + a^2b^n - a^2y^\alpha ab^n - a^2y^\alpha ab^n + a^2y^\alpha a^2y^\alpha b^n$

$$= a^2\,(y^\alpha + b^n - y^\alpha ab^n - y^\alpha ab^n + y^\alpha a^2y^\alpha b^n) = a^2z$$

where $z = y^\alpha + b^n - y^\alpha ab^n - y^\alpha ab^n + y^\alpha a^2y^\alpha b^n$. Therefore a is strongly regular element. □

In the following; For any ring R, let $P(R)$ be the prime radical of R and N be the set of the nilpotent elements of R. [4]

Theorem 2.7 :

Let R be a commutative ring, if $R/P(R)$ is strongly regular ring then for each $a\in R$ there exists a positive integer $n\neq 1$ such that a^n is strongly γ-regular element.

Proof: Since $R/P(R)$ is strongly regular ring, then for each $a+P\in R/P\,(R)$ there exists $y+P\in R/P\ (R)$ such that $a+P=(a+P)^2\,(y+P)=(a^2+P)(y+P)=a^2y+P$, then $a-a^2y\in P(R)$. So $a-a^2y\in N$. Hence there exists $n\in z^+$ such that $(a - a^2y)^n = 0$

Now, $(a-a^2y)^n=a^n-c_1^n a^{n+1} y+c_2^n a^{n+2} y^2-\ldots\ldots+(-1)^n a^{2n}y^n=0$, then $a^n=a^{n+1}z$, where $z=c_1^n y+c_2^n ay^2-\ldots\ldots+(-1)^n a^{n-1}y^n$ So $a^n=aa^nz=aa^{n+1}zz=a^{n+2}z^2=\ldots..=a^{2n}z^n$.
Therefore a^n is strongly γ-regular element. □

Definition 2.8: [7]

A ring *R* is said to be a semi-commutative ring if every idempotent element in *R* is central.
Hence every reduced ring is semi-commutative ring. [7].

Theorem 2.9:

Let *R* be a ring. If *R* is semi-commutative strongly γ-regular ring, then *R/N* is γ-regular ring.

Proof:

Since *R* is strongly γ-regular ring, then from (Theorem 5.6, [4]) *R* is γ-regular ring. So *R/N* is strongly γ-regular ring (Theorem 5.10, [4]).□

3. Strongly γ-Regular Rings With Condition (*)

The following condition (*) was introduced by Mohammad A. J. and Salih S. M. in [4].

(*): let *R* be a ring such that for every $1\neq a\in R$ *and* $b\in R$, there exist a positive integer m >1 such that $ab=b^ma$.

In this section we discus the connection between strongly γ-regular ring with the other rings which they are commutative, reduced or satisfies condition (*).

Theorem 3.1 :

Let *R* be a reduced ring. If *R* is strongly π-regular ring satisfies condition (*). Then *R* is strongly γ-regular ring.

Proof: Since *R* is strongly π-regular ring, then for every $a\in R$ there exists $m\in z^+$ and element $b\in R$ such that $a^m=a^{m+1}b$. Now since *R* satisfies condition (*), then for every $a,b\in R$, $ab=b^na$ for some positive integer $n>1$. Then $a^m=a^mb^na$. So $(1-b^na)\in r(a^m)$. By 4.8 [2], $r(a^m)=r(a)$. By [Lemma 2.4] $r(a)=l(a)$, whence $(1-b^na)a=0$. then $a=a^2b^n$. Therefore *R* is strongly γ-regular ring. □

Theorem 3.2 :

If *R* is a reduced ring satisfies condition (*), and for all $a\in R$ there exists unit element $d\in R$ and some idempotent $e\in R$ such that $a=de$. Then *R* is strongly γ-regular ring.

Proof: Let $a \in R$, and $a=de$ for some unit $d \in R$ and some idempotent $e \in R$. Hence $e=xa$, where x is the inverse of d. Now $ae=axa=dexa=dee=de^2=de=a$. Therefore $a=ae=axa$. Since R satisfies condition (*), then $ax=x^n a$ with a positive integer $n \neq 1$ for every $a,x \in R$. Then $a=axa=x^n aa=x^n a^2$. Therefore R is strongly γ-regular ring. □

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