The Relation Between the Alternating Group and Standard Row Young's Diagrams

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Received **26 / 12 / 2017**

Accepted **23 / 01 / 2018**

الخلاصة:

في هذا البحث قدمنا العلاقة بين الزمرة المتناوبة A_n ومخططات يونك في حالة الصفوف القياسية، فقد تم تقسيم العمل الى مرحلتين ... **المرحلة الاولى** هو ايجاد عدد التبديلات الزوجية بالاعتماد على مفهوم التجزئة μ من خلال انشاء خوارزمية لهذا الغرض، **والمرحلة الثانية** من خلال ايجاد علاقة تربط بين طول الدورة والتجزئة μ.

Abstract:

This study tackles the relationship between the alternating group A_n and young's diagrams concerning standard rows. The study has been divided into two stages… **First**, even permutations have been found depending on the conception of partition μ through formulating an algorithm for this purpose. **Secondly**, the relationship between the cycle length and partition μ has been found.

3 . 1 Introduction:

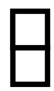
Let n be a non-negative integer, a composition μ of n is a sequence $\mu = (\mu_1 ,\mu_2,\cdots \mu_r)$ of non-negative integers such that $|\mu| = \sum_{j=1}^{r} \mu_j = n$, [5]. For example, if $n = 4$, the following sequences are compositions:

$(4), (3,1), (2,2), (1,3), (2,1,1), (1,2,1), (1,1,2), (1,1,1,1)$.

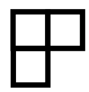
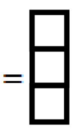
A composition μ is said to be a partition for n if $\mu_j \geq \mu_{j+1}$, [5]. In this case of the above mentioned example $n = 4$, the following sequences realize the condition of partition: $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$.
young's diagrams [5] for partition $\mu = (\mu_1, \mu_2, \cdots \mu_r)$ of n is :

$$[\mu] = \{(x, y):\ 1 \leq y \leq \mu_x\ ,\ x \geq 1\} \subseteq \mathrm{N} \times \mathrm{N}$$

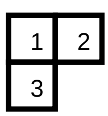
The elements of $[\mu]$ are called nodes for partition μ of n, and these nodes are elements from $\mathrm{N} \times \mathrm{N}$, it is represented by a diagram in the form of a system of adjacent square boxes, where μ_1 of squares are included in the upper row followed by μ_2 boxes in the row that follows and so on. for example, young diagrams for partition μ in case of $n = 2$ is

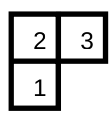
$\mu_1 = (2) =$  , $\mu_2 = (1,1) =$ 

In case of $n = 3$:

$\mu_1 = (3) =$  , $\mu_2 = (2,1) =$  , $\mu_3 = (1,1,1) =$ 

It is said that the rows of young diagram are standard if the numbers 1 to n are included in each row increasingly [7]. For example, the permutations realized in the case of partition (2,1) are only:



and


[1] If $f \in S_n$, then f is said to be an even (odd) permutation if and only if the multiplication output is:

$$\prod_{i>k} \frac{f(i) - f(k)}{i - k} = \begin{cases} 1\ then\ f\ is\ even \\ -1\ then\ f\ is\ odd \end{cases} \qquad \forall\, i, k = 1,2,3, \dots, n$$

For example, the permutation $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \in S_n$ is even because:

$$\prod_{i>k} \frac{f(i) - f(k)}{i - k} =$$

$$\left(\frac{f(4) - f(3)}{4 - 3}\right)\left(\frac{f(4) - f(2)}{4 - 2}\right)\left(\frac{f(4) - f(1)}{4 - 1}\right)\left(\frac{f(3) - f(2)}{3 - 2}\right)\left(\frac{f(3) - f(1)}{3 - 1}\right)$$

$$\left(\frac{f(2) - f(1)}{2 - 1}\right) = 1$$

The alternating group A_n is defined as the group of all even permutations on a finite group, and it is a subgroup of a symmetric group S_n with $\frac{n!}{2}$ elements. [6]

The following algorithm has been formulated to find the even cases in S_n.

Algorithm (1.1)

```
Begin
s ← size of group
h ←1
N ← factorial (s)
Prod ← 1
Repeat for x1=[1 to s]
{       Repeat for x2=[1 to s]
        a =[x1    x2]
        if (a has no equal elements)
        {   Repeat for x3=[1 to s]
                a=[x1   x2   x3]
                if (a has no equal elements)
                {   Repeat for x4=[1 to s]
                        a=[x1   x2   x3   x4]
                        if (a has no equal elements)
                        {   Repeat for x5=[1 to s]
                                {  a=[x1   x2   x3   x4   x5]
                                   if (a has no equal elements)
        {
                ⋮
                                            Repeat until:   a=[x1   x2   …  xs]
                                             if (a has no equal elements)
                                                {   Prod ← 1
                                                    Repeat for c=s to 1 steps(-1)
                                                  {      Repeat for d=c-1  to 1 steps (-1)
                                                      {prod ← prod * (a(c)−a(d))/(c−d)
                                                           }
                                                      }
                                                      if (prod >= 0)
                                                      add a to alt matrix at row h
                                                      h ← h+1
                                            }
```

```
                    }
               }
          }
     }
}
End
```

[2] The permutations that can be written as a cycle of length 2. Such cycles are called transpositions, that is:
$(a_1, a_2, \cdots, a_n) = (a_1 a_n)(a_1 a_{n-1}) \cdots (a_1 a_3)(a_1 a_2)$.

[2] A permutation of a finite set is even or odd according to weather it can be expressed as a product of an even number of transposition or the product of an odd number of transpositions.

The following facts can be used:

1) The product of two of even or odd permutations will be even permutation.
2) The product of odd and even permutation will be odd.(similarly if it is even permutation and odd permutation, it will be odd).

Example: permutations $f_1 = (15342)$, $f_2 = (23)(45) \in S_5$ are even and belong to $\mathrm{A_n}$ because:

$f_1 = (15342)$

$= (12)(14)(13)(15)$

Since the number of transpositions is even, therefore f_1 permutation is even.

$f_2 = (23)(45)$

It is obvious it is even permutation, it is also possible to benefit from the facts stated in the previous definition.

Note: the identity element is considered even because it could be written as the product of p transpositions, i.e

$id = \tau_1\ \tau_{2\ldots}\ \tau_p$ where p is even.

In this way it is possible to find all the even permutations in the alternating group $\mathrm{A_n}$. For example in

$\mathrm{A_2} = \frac{2!}{2} = 1$

$= \{(1)(2)\}$

That is, the identity element only

$\mathrm{A_3} = \frac{3!}{2} = 3$

$= \{(1)(2)(3), (123), (132)\}$

$$A_4 = \frac{4!}{2} = 12$$

$$= \begin{Bmatrix} (1)(2)(3)(4), (1)(234), (1)(243), (12)(34), (123)(4), (124)(3), \\ (132)(4), (134)(2), (13)(24), (142)(3), (143)(2), (14)(23) \end{Bmatrix}.$$

3 . 2 The Alternating Group and Young Diagram

In this section, we detected the number of even permutations from A_n which corresponds the partition μ to n relate to young diagrams in the case of standard rows through an algorithm for counting the number as well as showing all the cases for each partition. Here, two cases appear: the first is called general and the second will be tackled in specific cases as illustrated below:

Algorithm (2.1)

```
Begin
s ← size of group
h ←1
N ← factorial (s)
Prod ← 1
Repeat for x1=[1 to s]
{       Repeat for x2=[1 to s]
        a =[x1    x2]
        if (a has no equal elements)
        {       Repeat for x3=[1 to s]
                a=[x1   x2   x3]
                if (a has no equal elements)
                {       Repeat for x4=[1 to s]
                        a=[x1   x2   x3   x4]
                        if (a has no equal elements)
                        {       Repeat for x5=[1 to s]
                                {       a=[x1   x2   x3   x4   x5]
                                        if (a has no equal elements)
                                          {
⋮
                                                Repeat until:   a=[x1   x2   … xs]
                                                if (a has no equal elements)
                                               {          Prod ← 1

                                                Repeat for c=s to 1 steps(-1)
```

```
                                        {       Repeat for d=c-1  to 1 steps (-1)
                                                {       prod ← prod * (a(c)−a(d))/(c−d)
                                                }
                                        }
                                        if (prod >= 0)
                                        add a to alt matrix at row h
                                        h ← h+1
                              }
                         }
                    }
               }
          }
  }
```

// Set initial variable values:
AS ← 0, A(S-1)(1) ← 0, A(S-2)(2) ← 0, A(S-2)(1)(1) ← 0 … A(1)(1) … (1) ←0.

Repeat for [i= 1 to $\frac{\text{factorial }(s)}{2}$]

{
// Checking if rows that have values (1 to s) are sorted
if Alt (row i) is sorted
add Alt (row i) to ArS matrix at row AS
AS ← AS+1
// Checking if rows that have values (1 to s-1) are sorted
if Alt (row I, columns 1 to s-1) is sorted
add Alt (row i) to Ar(S-1)(1) matrix at row A(S-1)(1)
A(S-1)(1) ← A(S-1)(1)
// Checking if both rows that have values (1 to s-2) and (s-1 to s) are sorted
if Alt (row I, columns 1 to s-2) is sorted and Alt (row I, columns s-1 to s) is sorted
add Alt (row i) to Ar(S-2)(2) matrix at row A(S-2)(2)
A(S-2)(2) ← A(S-2)(2) +1
⋮
}
End

Proposition (2.2): the number of even permutations $\sigma(\mu)$ in A_n for any partitioning $\mu = (\mu_1, \mu_2, \cdots \mu_r)$ is:

$$\sigma(\mu) = \frac{n!}{2(\mu_1!\ \mu_2!\ \cdots \mu_r!)}$$

Proof: According to [7], the law in a symmetric group is:

$$M^{\mu} = \frac{n!}{\mu_1!\ \mu_2!\ \cdots \mu_r!}$$

Since the relationship between the symmetric and the alternating is $\frac{S_n}{2}$, therefore

$$\sigma(\mu) = \frac{n!}{2(\mu_1!\ \mu_2!\ \cdots \mu_r!)}$$

Proposition (2.3) exceptional cases

Building on using the general rule (2.2) and comparing the results that appeared with the computer program, certain exceptional cases are detected that rule (2.2) cannot be applied on it, because if it is used, it will not yield accurate results that result in correct solution and due to some reasons like the possibilities of computer and an electricity system in the country which does not allow the program to continue to work until the end without turn off. Hence we worked until A_7, so we could not find a suitable base to work on it. Therefore, we presented them as exceptional cases as follows:

(2.3.1) if n is odd; we will have the following cases:

a) $(n-1,1)$

b) $(2^q,1)$ where $q = 1,2,\ldots$

c) $(n-3,3)$

d) $(4,3,l)$ where $l = 0,1,2,\ldots$

e) $(z,2^q,1)$ where $z = 2l,\quad l \geq 0$ و $q = 1,2,\ldots$

(2.3.2) if n is even; we will have the following case:

(2^q) where $q = 1,2,\ldots$

(2.3.3) if n is even or odd, the following will be the case:

$(n-2,2)$

To illustrate this, we will take the cases A_2 to A_4, all other cases can be found in the computer program. A_n^* is used to indicate the realization of the alternating group conditions and the standard rows of young diagram as well.

$A_2^* =$

$\{\ (2) = 1 \longrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$ and $(1,1) = 1 \longrightarrow \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\ \}$

$A_3^* =$

$\{(3) = 1 \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$, $(2,1) = 2 \longrightarrow \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$

$(1,1,1) = 3 \longrightarrow \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$, $\begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline 2 \\ \hline\end{array}$ and $\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 1 \\ \hline\end{array}\ \}$

$A_4^* =$

$\{(4) = 1 \longrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}$. $(3,1) = 2 \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$,

$(2,2) = 4 \longrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline\end{array}$,

$(2,1,1) = 6 \longrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$,

$(1,1,1,1) = 12 \longrightarrow \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}\ \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}\ \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 4 \\ \hline 3 \\ \hline\end{array}\ \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline 1 \\ \hline 2 \\ \hline\end{array}\ \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline 2 \\ \hline\end{array}\ \begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\ \begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline\end{array}\ \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 1 \\ \hline 4 \\ \hline\end{array}\ \begin{array}{|c|}\hline 4 \\ \hline 1 \\ \hline 3 \\ \hline 2 \\ \hline\end{array}$

$\begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline 3 \\ \hline 1 \\ \hline\end{array}\ \begin{array}{|c|}\hline 4 \\ \hline 2 \\ \hline 1 \\ \hline 3 \\ \hline\end{array}$ and $\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 4 \\ \hline 1 \\ \hline\end{array}\ \}$

To count the number of even permutations in the partition (3,1) and (2,2) concerning the alternating group A_4 by using (2.1) and (2.2), the permutation will be:

$$\sigma(3,1) = \frac{4!}{2\ (3!\ 1!)}$$
$$= 2$$
$$\sigma(2,2) = \frac{4!}{2\ (2!\ 2!)}$$
$$= 3$$

As illustrated $\sigma(2,2)$ is considered one of the specific cases according to (2.3).

3 .3 Compute the Even Permutations by Using the Length of Cycles and Partition

First, we will refer to the concept of conjugation classes in general. The row of conjugate is defined as follows, if we have the two elements x,y in a group G, then $x = g^{-1}yg$ for each $g \in G$ is an equivalence relationship on G, hence the rows of equivalence are called rows of conjugates for G [3]. It's a fact that each two elements in S_n are conjugates if they have the same proposition or length of the cycle as A_n . The construct cycle represents a series of non- negative integers symbolized $\{u_r\}$ where $r = 1,2,\dots,i$, u_r represents all the non- interlinked symmetric cycles, where $\sum_{r=1}^{i} u_r = n$,[4]. The length of the cycle can be defined as follows, if we have $\in S_n$, we say f is a cycle and its length (K- cycle), $f = (n_1, n_2, \cdots, n_K)$ if $f(n_i) = n_{i+1}$ for $1 \le i < K$ and $f(n_K) = n_1$, also $f(n) = n$ when $n \ne n_1, n_2, \cdots, n_K$.[1]

As an illustration, we will provide all the lengths of the cycles in A_2 to A_4, and later we will provide tables showing the lengths and number of young diagrams that appropriate each length until A_7.

$$A_2 = \{(1)(2)\} \longrightarrow K(A_2) = 1^{\ 2}$$

$$A_3 = \{(1)(2)(3), (123), (132)\} \longrightarrow K(A_3) = \{1^3\ , 3\ , 3\}$$

This means we have only two types of conjugate rows in which the permutations are even and they are $3-$ cycle and $1-$ cycle

$$A_4 = \begin{Bmatrix} (1)(2)(3)(4), (1)(234), (1)(243), (123)(4), (124)(3), (132)(4), \\ (134)(2), (142)(3), (143)(2), (13)(24), (12)(34),, (14)(23) \end{Bmatrix} \longrightarrow$$

$$K(A_4) = \{1^4\ , 13\ , 13, 13, 13, 13, 13, 13, 13, 2^2\}$$

Here in the case of A_4 we have three types of conjugate rows in which the permutations are even and they are $(3-$ cycle$)(1-$ cycle$)$, $(1-$ cycle$)$ and $(2-$cycle$)(2-$ cycle$)$.

Now we will offer some rules that account for the number of even permutations in A_n.

Rule (3.1): If n is odd, the cycle length $n=(r)$, however if n is even, the cycle length is $n=(1)(r-1)$ for any partition $\mu=(\mu_1,\mu_2,\cdots\mu_r)$, the number of even permutations that are symbolized β_1 will be:

$$\beta_1=\begin{cases}\dfrac{n!}{(r-1)!\mu_1!\ \mu_2!\cdots\mu_r!} & if\ n\ is\ even\\[2ex] \dfrac{n!}{(r)!\mu_1!\ \mu_2!\cdots\mu_r!} & if\ n\ is\ odd\end{cases}$$

This use is illustrated in the following tables, the last column in each table.

Rule (3.2): In the case of partition $(1,1,\ldots,1_r)$ and all case of length indicated in white color in the last row of each table in the following tables, the relationship will be symbolized β_2 as pointed out by [4] as in:

$$\beta_2=\frac{n!}{\prod_r r^{u_r}\ u_r!}$$

Rule (3.3): Here some exceptional cases appeared as it is the case in the second section that will displayed in details. This case has been treated by using even permutations in A_2 till A_7 as follows:

(3.3.1) if $\mu=(n)$, the standard rows illustrated in the following tables in **yellow** will be according to the length of the cycle as follows:

$$\begin{cases}1 & if\quad (1)^n\\ 0 & if\quad 1^{(n-4)}\,2^2\quad or\quad 1^{(n-3)}\,3\end{cases}$$

(3.3.2) if $\mu=(r,1)$, the standard rows in **light orange** in the following tables will be according to the cycle length as follows:

$$\begin{cases}1 & if\quad (1)^n\quad or\quad 1^{(r-2)}\,3\\ 0 & if\quad 1^{(r-3)}\,2^2\end{cases}$$

(3.3.3) if $\mu = (r, 2)$, the standard rows in **light blue** in the following tables will be according to the cycle length as follows:

$$\begin{cases} 1 & if \quad (1)^n \quad or \quad 1^{(r-2)}\, 2^2 \\ 2 & if \quad 1^{(r-1)}\, 3 \end{cases}$$

(3.3.4) if $\mu = (r, 3)$, the standard rows in **dark green** in the following tables will be according to the cycle length as follows:

$$\begin{cases} 1 & if \quad (1)^n \quad or \quad 1^{(r-1)}\, 2^2 \\ 2 & if \quad 1^{(r)}\, 3 \end{cases}$$

(3.3.5) if $\mu = (r, 1,1)$, the standard rows in **light green** in the following tables will be according to the cycle length as follows:

$$\begin{cases} 1 & if \quad (1)^n \quad or \quad 1^{(r-2)}\, 2^2 \\ 4 & if \quad 1^{(r-1)}\, 3 \end{cases}$$

(3.3.6) if $\mu = (r, 2,1)$, the standard rows in **light purple** in the following tables will be according to the cycle length as follows:

$$\begin{cases} 1 & if \quad (1)^n \\ 3 & if \quad 1^{(r-1)}\, 2^2 \\ 6 & if \quad 1^{(r)}\, 3 \end{cases}$$

(3.3.7) if $\mu = (r, 3,1)$, the standard rows in **light brown** in the following tables will be according to the cycle length as follows:

$$\begin{cases} 1 & if \quad (1)^n \\ 3 & if \quad 1^{(r)}\, 2^2 \\ 6 & if \quad 1^{(r+1)}\, 3 \end{cases}$$

(3.3.8) if $\mu = (r, 1,1,1)$, the standard rows in **dark orange** in the following tables will be according to the cycle length as follows:

$$\begin{cases} 1 & if \quad (1)^n \\ 6 & if \quad 1^{(r-1)}\, 2^2 \\ 11 & if \quad 1^{(r)}\, 3^2 \end{cases}$$

(3.3.9) if $\mu = (r, 2,2)$, the standard rows in **dark blue** in the following tables will be according to the cycle length as follows:

$$\begin{cases} 1 & if \quad (1)^n \\ 6 & if \quad 1^{(r)}\, 2^2 \\ 8 & if \quad 1^{(r+1)}\, 3 \end{cases}$$

(3.3.10) if $\mu = (r, 2,1,1)$, the standard rows in **dark purple** in the following tables will be according to the cycle length as follows:

$$\begin{cases} 1 & if \quad (1)^n \\ 10 & if \quad 1^{(r)}\, 2^2 \\ 14 & if \quad 1^{(r+1)}\, 3 \end{cases}$$

(3.3.11) if $\mu = (r, 1,1,1,1)$, the standard rows in **red** in the following tables will be according to the cycle length as follows:

$$\begin{cases} 1 & if \quad (1)^n \\ 21 & if \quad 1^{(r)}\, 2^2 \\ 24 & if \quad 1^{(r+1)}\, 3 \end{cases}$$

(3.3.12) if $\mu = (r, 2,2,1)$, the standard rows in **dark grey** in the following tables will be according to the cycle length as follows:

$$\begin{cases} 1 & if \quad (1)^n \\ 15 & if \quad 1^{(r)}\, 2^2 \\ 17 & if \quad 1^{(r+1)}\, 3 \end{cases}$$

(3.3.13) if $\mu = (r, 2,1,1,1)$, the standard rows in **dark brown** in the following tables will be according to the cycle length as follows:

$$\begin{cases} 1 & if \quad (1)^n \\ 28 & if \quad 1^{(r)}\, 2^2 \quad or \quad 1^{(r+1)}\, 3 \end{cases}$$

(3.3.14) if $\mu = (r, 1,1,1,1,1)$, the standard rows in **pink** in the following tables will be according to the cycle length as follows:

$$\begin{cases} 1 & if \quad (1)^n \\ 55 & if \quad 1^{(r+1)}\, 2^2 \\ 45 & if \quad 1^{(r+2)}\, 3 \end{cases}$$

Hereafter, the tables A_2 to A_7 are sequenced where all the previously mentioned cases are illustrated, each case is given a specific color:

A_2	
Cycle type / Partition	1^2
(2)	1
(1,1)	1

(3.4)

A_3		
Cycle type / partition	1^3	3
(3)	1	0
(2,1)	1	1
(1,1,1)	1	2

(3.5)

A_4			
Cycle type / partition	1^4	2^2	13
(4)	1	0	0
(3,1)	1	0	1
(2,2)	1	1	2
(2,1,1)	1	1	4
(1,1,1,1)	1	3	8

(3.6)

A_5				
Cycle type / partition	1^5	12^2	$1^2 3$	5
(5)	1	0	0	0
(4,1)	1	0	1	1
(3,2)	1	1	2	2
(3,1,1)	1	1	4	4
(2,2,1)	1	3	6	6
(2,1,1,1)	1	6	11	12
(1,1,1,1,1)	1	15	20	24

(3.7)

A_6						
Cycle type / partition	1^6	$1^2 2^2$	$1^3 3$	24	3^2	15
(6)	1	0	0	0	0	0
(5,1)	1	0	1	0	0	1
(4,2)	1	1	2	1	1	3
(4,1,1)	1	1	4	2	1	6
(3,2,1)	1	3	6	5	3	12
(3,1,1,1)	1	6	11	12	6	24
(3,3)	1	1	2	1	1	4
(2,2,2)	1	6	8	9	6	18
(2,2,1,1)	1	10	14	19	10	36
(2,1,1,1،1)	1	21	24	42	20	72
(1,1,1,1,1،1)	1	45	40	90	40	144

(3.8)

A_7								
Cycle type / partition	1^7	$1^3 2^2$	$1^4 3$	$2^2 3$	$3^2 1$	421	51^2	7
(7)	1	0	0	0	0	0	0	0
(6,1)	1	0	1	0	0	0	1	1
(5,2)	1	1	2	0	1	1	3	3
(5,1,1)	1	1	4	0	1	2	6	6
(4,2,1)	1	3	6	2	5	8	14	15
(4,1,1,1)	1	6	11	3	9	18	27	30
(4,3)	1	1	2	1	2	2	5	5
(3,2,2)	1	6	8	6	12	19	26	30
(3,2,1,1)	1	10	14	12	22	41	50	60
)1(3,1,1,1،	1	21	24	24	44	90	96	120
)1(3,3،	1	3	6	4	7	11	18	20
(2,2,2,1)	1	15	17	21	36	66	72	90
(2,2,1,1,1)	1	28	28	44	69	142	138	180
1,1,1,1),1,(2	1	55	45	94	140	301	264	360
1,1,1,1),1,(1,1	1	105	70	210	280	630	504	720

(3.9)

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