On Almost WJCP-Injective Rings

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حول الحلقات الغامرة من النمط – WJCP تقريباً

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الخلاصة

يقال للحلقة R بأنها غامرة من النمط – WJCP تقريباً إذا كان لكل (مثالي ايمن غير منفرد) $b \notin Y(R)$، يوجد مثالي أيسر في R بحيث أن $l_R r_R(b) = Rb \oplus X_b$. في هذا البحث أعطينا مميزات وخواص الحلقات الغامرة من النمط – WJCP تقريباً والذي هو تعميم للحلقات الغامرة من النمط – WJCP والغامرة من النمط – AP تقريباً. كذلك درسنا انتظامية الحلقات الغامرة من النمط –WJCP تقريباً اليمنى وتوسيع بعض النتائج المعروفة في الحلقات الغامرة من النمط – WJCP اليمنى إلى الحلقات الغامرة من النمط – WJCP تقريباً اليمنى.

ABSTRACT

Let R be a ring. The ring R is called right almost WJCP-injective. If for any $b \notin Y(R)$, (right non singular) there exists a left ideal X_b of R such that $l_R r_R(b) = Rb \oplus X_b$. In this paper, we give some characterization and properties of almost WJCP-injective rings, which are proper generalization of JCP-injective ring and almost AP-injective ring. Then, we study the regularity of the right almost WJCP-injective ring and some important results which are known for the right JCP-injective rings to be hold for the right almost WJCP-injective rings.

Keywords: JCP-injective rings, almost nil-injective rings, quasi regular rings and reduced rings.

1- Introduction

In this paper, R will be an associative ring with identity and all modules are unitary right R-modules. For subset X of R, the right (left) annihilator of X in R is denoted by $r(X)$ $(l(X))$. If $X=\{a\}$, we usually abbreviate $r(a)$ and $l(a)$for any a $\in$ R. We write $J(R)$, $N(R)$, $Y(R)$, $(Z(R))$ for the Jacobson radical, the set of nilpotent elements, and right (left) singular ideal of R, respectively.

At first, we recall that a ring R is called right principally injective [4] (or P-injective for short), if every homomorphism from a principally right ideal of R to R can be extended to an endomorphism of R, or equivalently $l\big(r(a)\big)=Ra$ for all $a\epsilon R$. The notion of the right P-injective rings has been generalized by many authors see ([2], [5]). In ([12], [14]) the right P-injective rings are almost principally injective rings, a ring R is said to be almost principally injective (or AP-injective for short), if for any $a\epsilon R$, there exists a left ideal X_a such that $l(r(a))=Ra\oplus X_a$.

Von Neumann regular rings have been studied extensively by many authors (for example [3]). It is well known that a ring R is regular if for any $a\epsilon R$, there exists $b\epsilon R$ such that $a=aba$.

In [7] JCP-injective rings are studied. A ring R is called a right JCP-injective, if for any right nonsingular element c of R and any R-homomorphism $g: cR\to R$, there exists $m\epsilon R$ such that $g(ca)=mca$ for all $a\epsilon R$. Cleary, the right P-injective rings are right JCP-injective rings. The nice structure of JCP-injective rings draws our attention to define almost WJCP-injective rings (or the right AWJCP-injective rings), and to investigate their characterizations and properties.

A ring R is called reduced, if $N(R)=0$. A ring R is said to be a biregular ring if for any $a\epsilon R$, RaR is generated by central idempotent [11]. In [8] the module M is called almost principally a small injective (or APS-injective for short) if for any $a\epsilon J(R)$, there exists an S-submodule X_a of M such that $l_M(r(a))=Ma\oplus X_a$ as left S-module. If R_R is an APS-injective module, then we call R a right APS-injective.

2- Characterizations of Right AWJCP-Injective Rings

In this section, we shall study characterizations of right AWJCP-injective rings.

Definition 2.1

Let M_R be a module with $S=End(M_R)$. The module M is called right almost WJCP-injective (AWJCP-injective), if for any $a\epsilon R\backslash Y(R)$, there exists an S-submodule X_a of M such that $l_M r_R(a)=Ma\oplus X_a$ as left S-module. If

R_R is almost WJCP-injective, then we call R_R a right almost WJCP-injective ring.

Every AP-injective ring is AWJCP-injective but the converse is not true. [Example 2.4, 7]

Theorem 2.2

Let $\{X_a : a\epsilon R$ be an index of left ideal$\}$, then the following are equivalent:

1) If $0 \neq a \not\in Y(R)$, then $l(r(a)) = Ra \oplus X_a$.
2) If $k \notin Y(R)$, $a\epsilon R$, then $l\big(aR \cap r(k)\big) = (X_{ka})_l + Rk$ with $ka\epsilon R\backslash Y(R)$ and $(X_{ka}\text{: a})_l \cap Rk \subseteq l(a)$ for all $a\epsilon R$, where $(X_{ka}: a)_l = l(aR)$ if $ka = 0$.

Proof:

The proof is similar to that of (Lemma 3.1, [5]) ■

An element $a\epsilon R$ is called a right regular if $r(a) = 0$ [8].

Theorem 2.3

Let R is a right AWJCP-injective. Then:

1) Any right regular element of R is left invertible.
2) $Y(R) \subseteq J(R)$.
3) If P is a reduced principal right ideal of R, then $P = eR$ where $e^2 = e\epsilon R$ and $(1-e)R$ is an ideal of R.

Proof:

1) Let $0 \neq a\epsilon R$ such that $r(a) = 0$. Then $a \notin Y(R)$ and so $l(r(a)) = Ra \oplus X_a$ where X_a is a left ideal of R (R is AWJCP-injective). Hence $R = l(0) = Ra \oplus X_a$ since $r(a) = 0$, thus there exists $r\epsilon R$, $x\epsilon X_a$ such that $1 = ra + x$, $a = ara + ax$, $a(1-ra) = ax\epsilon Ra \cap X_a = 0$ so $(1-ra)\epsilon r(a) = 0$. Therefore $Ra = R$ and hence a is a left invertible.
2) If $y\epsilon Y(R)$ and $a\epsilon R$, then, $r(1-ay) = 0$ implies that $v(1-ay) = 1$ for some $v\epsilon R$ by (1). Hence $y\epsilon J(R)$.
3) Let P be a nonzero reduced principally a right ideal. Then $P = aR$ for some $a\epsilon R$, since $a^2 \notin Y(R)$, $l\,(r(a^2)) = Ra^2 \oplus X_{a^2}$ for some a left ideal X_{a^2} of R. Hence $r(a) = r(a^2)$ shows that $Ra \oplus X_a = l\big(r(a)\big) = l\big(r(a^2)\big) = R\,a^2 \oplus X_{a^2}$, $X_{a^2} \subseteq {}_R R$. Then there exists $r\epsilon R$, $x \in X_{a^2}$ such that $a = r\,a^2 + x$, $a^2 = ara^2 + ax$, $ax = (1-ar)a^2 \epsilon \mathrm{R}a^2 \cap X_{a^2} = 0$, $a^2 = ara^2$, $(1-ar)\epsilon l(a^2) = l(a) = 0$, which implies that $a = ara$ (P is reduced), where P is generated by the idempotent $e = ar$. Also for any $b\epsilon R$, $(\mathrm{eb} - \mathrm{ebe})^2 = 0$ implies $b = ebe$, where $eR(1-e) = 0$. Therefore $R(1-e) \subseteq (1-e)R$. ■

Lemma 2.4 [8]

Let R be a right APS-injective ring, then $J(R) \subseteq Y(R)$. ■

The following corollary follows immediately form Lemma 2.4 and Theorem 2.3.

Corollary 2.5

If R is a right AWJCP-injective and right APS-injective ring, then $J(R) = Y(R)$. ■

3. Regularity of Right AWJCP-injective Rings

A ring R is called PP, if for any $a \epsilon$R, aR is projective and R is a right SPP, if for any $a \notin Y(R)$, aR is projective. Every PP ring is SPP. A ring R is called quasi regular, if $a \epsilon aRa$ for all $a \notin Y(R)$[7].

A ring R is called a strongly regular, if for every $a \in R$ there exists $b \in R$ such that $a = a^2 b$. [10].

Remark 1: [7]

R is regular if and only if R is a right nonsingular and a right quasi regular.

Proposition 3.1

The following conditions are equivalent for a ring R:

1) R is a quasi regular ring.
2) R is a right JCP-injective and a right SPP ring .
3) R is a right AWJCP-injective and a right SPP ring.

Proof:

Obviously: 1 → 2 → 3.

3 → 1, Suppose that R is a right AWJCP-injective and right SPP-ring. For any $0 \neq a \notin Y(R)$, there exists a left ideal $X_a$of R such that $lr(a) = Ra \oplus X_a$. Since R is a right SPP, then $r(a) = eR$ with $e^2 = e \in R$. Let $f = 1 - e$. Then $lr(a) = Rf$, and $f^2 = f \in R$, and so $a = af$ and $f = da + x$ for some $d \in R$ and $x \in X_a$.

Thus $af = ada + ax$, $a - ada = ax \in Ra \cap X_a = 0$, this shows that $a = ada$, and so R is aquasi regular. ■

Corollary 3.2

Let R be a ring. Then R is a regular ring if and only if R is a right nonsingular,a right SPP, and an AWJCP-injective ring.

Proof:

It Follows from Proposition 3.1 and Remark 1.

Lemma 3.3 [14]

Suppose M is a right R-module with $S = End(M_R)$.

If $l_M(r_R(a)) = Ma \oplus X_a$, where X_a is a left S-submodule of M_R. Set $f: aR \to M$ a right R-homomorphism, then $f(a) = ma + x$ with $m \epsilon M$, $x \epsilon X_a$.

A right R-module M is called almost nil-injective [13], if for any $k \epsilon N(R)$, there exists an S-submodule X_k of M such that $l_M r_R(k) = Mk \oplus X_k$ as left S-module ($S = End(M)$). If R_R is almost nil-injective, then we call R a right almost nil-injective ring.

Theorem 3.4
Let R be a right SPP ring. Then R is a right AP-injective ring if and only if R is a right AWJCP-injective and every simple singular right R-module is almost nil-injective.
Proof:
First, we show that $Y(R) = 0$. Suppose that $Y(R) \neq 0$, then there exists $0 \neq b\epsilon Y(R)$ such that $b^2 = 0$. We claim that $Y(R) + r(b) = R$. Otherwise, there exists a maximal right essential ideal M of R such that $Y(R) + r(b) \subseteq M$. Thus R/M is almost nil-injective and $l_{R/M}\, r_R(b) = (R/M)b \oplus X_b$, for some a left ideal X_b of R/M. Let $f: bR \rightarrow R/M$ be defined by $f(br) = r + M$. Then f is well defined R-homomorphism so there exists $r\epsilon R$, $x\epsilon X_b$ such that
$1 + M = rb + M + x$ (Lemma 3.3), $1 - rb + M = x\epsilon (R/M)b \cap X_b = 0$. Hence $1 - rb\epsilon M$. Since $rb \in Y(R) \subseteq M$, then $1\epsilon M$, which is a contradiction. Therefore $Y(R) + r(b) = R$. Hence $1 = c + d$ for some $c\epsilon Y(R)$ and $d\epsilon r(b)$. Thus $b = bc$, $b(1 - c) = 0$. Since $c\epsilon Y(R) \subseteq J(R)$ [Theorem 2.3 (2)], $(1 - c)$ is invertible. Thus $b = 0$, which is a contradiction. Hence $Y(R) = 0$. By Corollary 3.2 R is a right AP-injective.
The converse is clear. ■
Lemma 3.5 [5]
If R is a right AP-injective ring, then $J(R) = Y(R)$. ■
By Theorem 3.4 and Lemma 3.5 we get:
Corollary 3.6
Let R be a right SPP ring. If R is a right AWJCP-injective and every simple singular right R-module is almost nil-injective, then $Y(R) = J(R) = 0$.
Theorem 3.7
Let R be a right AWJCP-injective ring and right PP. Then R is regular.
Proof:
Let $0 \neq a\epsilon R$. Then $a \notin Y(R)$ [Theorem 2.9, 7]. Since R is a right AWJCP-injective, then $l_R\, r_R(a) = Ra \oplus X_a$ for some left ideal X_a of R. Since R is a right PP-ring $r(a) = r(e)$, $e^2 = e\epsilon R$.
Thus $Re = lr(e) = lr(a) = Ra \oplus X_a$. Therefore $e = ba + x$ for some $x\epsilon X_a$ and $b\epsilon R$. So $a = ae = aba + ax$, $(1 - ab)a = ax\epsilon Ra \cap X_a = 0$, and $a = aba$. Hence R is regular. ■
Following [1], a ring R is called a left pseudo-morphic if for all $a\epsilon R$ there exists $b\epsilon R$ such that $Ra = l(b)$. Every regular rings is pseudo-morghic.
From Theorem 3.7 have:
Corollary 3.8
Let R be a right AWJCP-injective ring and a right PP. Then R is a left pseudo-morphic.
A ring R is called a left N duo, if Ra is an ideal of R for all $a\epsilon N(R)$ [6].

Lemma 3.9 [6]

1) Let R be a semiprime left N duo ring. Then R is reduced.

2) If R is a reduced, then $Y(R) = Z(R) = 0$.

Proposition 3.10

Let R be a semiprime left N duo ring, every simple singular right R-module is AWJCP-injective. Then R is a biregular ring.

Proof:

For any $a \epsilon R, l(RaR) = r(RaR) = r(a) = l(a)$. If $RaR \oplus r(a) \neq R$, then there exists a maximal right ideal M of R such that $RaR \oplus r(a) \subseteq M$. If M is not essential in R, then $M = r(e),\ e^2 = e \epsilon R$. Therefore $ea = 0$. Since R is a reduced $ae = 0$. Hence $e \epsilon r(a) \subseteq r(e)$, which is a contradiction. So M is essential in R. Since R is a reduced (Lemma 3.9) $Y(R) = 0$. Thus R/M is AWJCP-injective, then $l_{R/M}\, r_R(a) = (R/M)a \oplus X_a,\ X_a \subseteq R/M$.

Let $f\colon aR \to R/M$ be defined by $f(ar) = r + M$. Note That f is well defined.

So $1 + M = f(a) = ca + M + x,\quad c \epsilon R,\quad x \epsilon X_a,\ 1 - ca + M = x \epsilon R/M \cap X_a = 0$, $1 - ca \epsilon M$. Since $ca \epsilon RaR \subseteq M$, $1 \epsilon M$, which is a contradiction. Hence $RaR \oplus r(a) = R$ and so $RaR = eR, e^2 = e \epsilon R$. Since R is an abelian ring, R is a biregular ring. ■

R is called a right CAM-ring, if for any maximal essential right ideal M of R (if it exists) and for any right subideal I of M which is either a complement right subideal of M or a right annihilator ideal in R, I is an ideal of M [10].

The right CAM-rings generalize semismple artinian. [10]

In [10] , shows that semiprime right CAM-ring R is either a semisiple artinian or a reduced.

A ring is called right MERT ring, if every maximal essential right ideal M of R is an ideal of R. [6]

The Following theorem is generalization of [Theorem 5.8, 7]

Theorem 3.11

The following are equivalent for a ring R which is SPP

1) R is either a semisimple artinain or a strongly regular ring.

2) R is a semiprime, a right AWJCP-injective, a right CAM-ring.

3) R is a semiprime, a right CAM-ring, a MERT ring every simple singular right R-module is AWJCP-injective.

Proof:

1→i , i=2, 3 are obvious.

2→1, if R is not a semiprime artinian ring, then R is reduced. By Corollary 3.2 , R is a regular ring. Therefore R is a strongly regular ring.

3→1, if R is not semisimple artinain ring, then R is reduced. Let $0 \neq a\ \epsilon R$. If $aR \oplus r(a) \neq R$. Then $aR \oplus r(a) \subseteq M$ for some maximal essential right ideal

M of R. Since R is a reduced, then $Y(R) = 0$. By a assumption, then simple singular right R-module R/M is AWJCP-injective,

thus $l_{R/M}r_R(a) = (R/M)a \oplus X_a$, $X_a \subseteq R/M$. Let $f: aR \rightarrow R/M$ be defined by $f(ar) = r + M$.

Note that f is well defined. Thus there exists $c \in R, x \in X_a$, such that $1 + M = f(a) = ca + M + x$,

then $1 - ca + M = x \in R/M \cap X_a = 0, 1 - ca \in M$. But $ca \in M$ then $1 \in M$, because R is a MERT ring and M is an ideal. It is a contradiction. Hence $aR \oplus r(a) = R$ and then R is a strongly regular ring. ■

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