Automatic Continuity of Some Types of Double Derivations on Semisimple Banach Algebras

Amir A. Mohammed and Lamia K. Ismail
Department of Mathematics, College of Education of Pure Science University of Mosul, Mosul, Iraq
lamiak.ismial88@gmail.com

Received	Accepted
15 /01/2018	06/03/2018

الخلاصة :

تبعا لـ بيينا في [9] وعلي ومحمد في [4]، قدمنا المشتقة الثنائية من النمط $(g, h) - c -$ ومن النمط $(g, h) - c -$ المعممة على جبر باناخ معقد شبة بسيط، منطلقهما مثالية أساسية ليست بالضرورة مغلقة و أثبتنا أنهما تكونان قابلتان للانغلاق. كحالة خاصة، أثبتنا كل مشتقة ثنائية من النمط $(g, h) - c -$ و من النمط $(g, h) - c -$ المعممة على أية مثالية غير صفرية للجبر $- C^*$ الأولي تكونان مستمرتان.

Abstract

Following Villena in [9] and Mohammed and Ali in [4], we introduce partially defined (g, h) - c - double derivation and generalized (g, h) - c - double derivation on a semisimple complex Banach algebra whose domain is not necessarily closed, essential ideal and we prove that they are closable. In particular, we show that every (g, h)- c -double derivation and generalized (g, h) - c - double derivation defined on any nonzero ideal of a prime C^*- algebra are continuous.

Keywords: automatic continuity, double derivation, ultraprimness, sliding hump sequence.

0. Introduction

Throughout this paper, $\mathcal{A}$ is a semisimple Banach algebra over complex field and $g, h : \mathcal{A} \to \mathcal{A}$ are linear mappings. If g and h are the identity maps and if $\mathcal{A}$ with or without identity we may conclude that g and h are continuous by Johnson and Sinclair in [1]. As a consequence, we can assume that g and h are continuous. So, we defined our derivation in this paper as in [5] and [7] as follows : A linear map $D_1 : \mathcal{A} \to \mathcal{A}$ is said to be (g, h) - c - double derivation on $\mathcal{A}$ if $D_1(a\,b) = D_1(a)\,b + a\,D_1(b) + g(a)\,h(b) + h(a)\,g(b)$, $\forall\, a, b \in \mathcal{A}$. Similarly, we defined our derivation in this paper as in [8] as follows:

A linear map $D_2 : \mathcal{A} \to \mathcal{A}$ is called generalized (g, h) - c - double derivation on $\mathcal{A}$ if there exists (g, h) - c - double derivation $D_1 : \mathcal{A} \to \mathcal{A}$ such that $D_2(a\,b) = D_2(a)\,b + a\,D_1(b) + g(a)\,h(b) + h(a)\,g(b)$, $\forall a, b \in \mathcal{A}$. Recall that, a nonzero ideal I of $\mathcal{A}$ is called essential if for any nonzero ideal J of $\mathcal{A}$ we have $I \cap J \neq \{0\}$. Note that, if $\mathcal{A}$ is prime then any nonzero ideal of $\mathcal{A}$ is essential. By essential defined (g, h) - c - double derivation we mean a linear map $D_1 : I \to \mathcal{A}$ such that I is essential and for all $a, b \in I$, $D_1(a\;b) = D_1(a)\;b + a\,D_1(b) + g(a)\,h(b) + h(a)\,g(b)$. Clearly if g or h or both are the zero maps then D_1 is the usual derivation, so (g,h) - c - double derivation is a generalization of derivation. Similarly, by essential defined generalized (g, h) - c - double derivation we mean a linear map $D_2 : I \to \mathcal{A}$ such that I is essential and for all $a,b \in I$, $D_2(a\,b) = D_2(a)\;b + a\,D_1(b) + g(a)\,h(b) + h(a)\,g(b)$.

Clearly if g or h or both are the zero maps and $D_1 = D_2$, then D_2 is the usual derivation, so generalized (g, h) - c - double derivation is a generalization of derivation. Also if $D_1 = D_2$, then generalized (g, h) - c - double derivation is (g, h) - c - double derivation.

Automatic continuity of derivations are studied by many researcher, we mention some of them of our present work see [1], [2], [5], [6] and [7].

In this paper, we will follow the same lines of [4] and [9]. We will use $D = D_1$ or D_2 when the results are true for both D_1 and D_2, otherwise we will use only D_1 or D_2.

Let $\mathcal{P}$ denote the set of primitive ideals P of $\mathcal{A}$ such that $I \not\subset P$. The primitive ideal P can be obtained as the kernel of a continuous irreducible representation of $\mathcal{A}$ on a complex Banach

space X_P, actually the irreducible representation of $\mathcal{A}$ is defined by the following mappings:
$\varphi : \mathcal{A} \longrightarrow BL(X_P)$ defined by $\varphi(a) = L_a$ and $L_a : X_P \longrightarrow X_P$ defined by $L_a(x) = ax$ and the $ker(\varphi) = P$ satisfying $\| a\, x \| \leq \| a \| \, \| x \|$, for all $a \in \mathcal{A},\ x \in X_P$.

Recall that the separating subspace $S(D)$ of D is defined to be the set of those a in $\mathcal{A}$ for which there is a sequence $\{ a_n \}$ in $\mathcal{A}$ with $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} D(a_n) = a$. It is well known that D is closable if and only if $S(D) = 0$, and it is easy to show that $I\, S(D) + S(D)\, I \subset S(D)$.
Let $\mathcal{P}_c = \{ P \in \mathcal{P} : S(D) \subset P \}$ and $\mathcal{P}_E = \{ P \in \mathcal{P} : S(D) \not\subset P \}$.
Note that $S(D) \subset \cap_{P \in \mathcal{P}_c} P = P_c$. We will show that D is closed if $P_c = 0$.

1. Main Results

We begin this section by the following fundamental results :

Proposition 1 : [9]

Let $P \in \mathcal{P}$ and J any non necessarily closed ideal of $\mathcal{A}$ satisfying $J \not\subset P$. Then one of the following assertions holds :

1) *The ideal of those elements $b \in J$ with $\dim bX_P < \infty$ acts irreducibly on X_P. Accordingly, given x , $y \in X_P$ with $x \neq 0$ there is $b \in J$ with $\dim b\, X_P = 1$ and $b\, x = y$.*
2) *There exist sequences $\{ b_n \}$ in J and $\{ x_n \}$ in X_P satisfying $b_n \ldots b_1 x_n \neq 0$ and $b_{n+1} \ldots b_1 x_n = 0$ for every $n \in \mathbb{N}$.*

Proof : see [9, lemma 1]

Let $\{ P_n \}$ be a sequence in $\mathcal{P}$. A sequence $\{ b_n \}$ in I is said to be a sliding hump sequence for $\{ P_n \}$ if for every $n \in \mathbb{N}$ there exists $x_n \in X_{P_n}$ such that $b_n \ldots b_1 x_n \neq 0$ and $b_{n+1} \ldots b_1 x_n = 0$ (see [9]).

Proposition 2 :

If there exists a sliding hump sequence for a sequence $\{ P_n \}$ in $\mathcal{P}$, then there is a natural number n for which

i) $S(D_1) \subset P_n$. In particular, $S(D_1) \subset P$ if $P_n = P$ for every $n \in \mathbb{N}$.
ii) $S(D_2) \subset P_n$. In particular, $S(D_2) \subset P$ if $P_n = P$ for every $n \in \mathbb{N}$.

Proof :

Let $\{ b_n \}$ be a sliding hump sequence for $\{ P_n \}$ then for every $n \in \mathbb{N}$, there exists $x_n \in X_{P_n}$ such that $b_n \ldots b_1 x_n \neq 0$ and $b_{n+1} \ldots b_1 x_n = 0$.

We can certainly assume that $\| b_n \| = \| g \| = \| h \| = \| x_n \| = 1$ for every $n \in \mathbb{N}$. We claim that there exist $n \in \mathbb{N}$ and a nonzero $x \in X_{P_n}$, such that the map $a \longmapsto D(a)x$ from I into X_{P_n} is continuous. If the claim fails, then all the maps $a \longmapsto D(a)\, b_n \ldots b_1\, x_n$ from I into X_{P_n} are discontinuous and we can construct inductively a sequence $\{ a_n \}$ in I satisfying :

$$\| D(a_n)\, b_n \ldots b_1\, x_n \| \geq n + \| \textstyle\sum_{k=1}^{n-1} D(a_k\, b_k \ldots b_1)\, x_n \| + \| D(c_{n+1})\, b_{n+1} \ldots b_1\, x_n \| \quad \ldots\ldots (1)$$

and $\| a_n \| \leq 2^{-n} \min \{ (1 + \| D_1(b_k \ldots b_1) \|)^{-1} : k = 1, \ldots, n \}$.

Now, we consider the element $c \in \mathcal{A}$ given by $c = \sum_{n=1}^{\infty} a_n\, b_n \ldots b_1$ and for every $n \in \mathbb{N}$, we write $c_n = a_n + \sum_{k=n+1}^{\infty} a_k\, b_k \ldots b_{n+1}$. Now we will follow the same way of [4] and [9], then we have $c = \sum_{k=1}^{n-1} a_k\, b_k \ldots b_1 + a_n\, b_n \ldots b_1 + c_{n+1}\, b_{n+1} \ldots b_1$.

Currently, we will prove the first part of this proposition :

(i) $$D_1(c) = \textstyle\sum_{k=1}^{n-1} D_1(a_k b_k \ldots b_1) + D_1(a_n)\, b_n \ldots b_1 + a_n\, D_1(b_n \ldots b_1) + g(a_n)\, h(b_n \ldots b_1) + h(a_n) g(b_n \ldots b_1) + D_1(c_{n+1}) b_{n+1} \ldots b_1 + c_{n+1}\, D_1(b_{n+1} \ldots b_1) + g(c_{n+1})\, h(b_{n+1} \ldots b_1) + h(c_{n+1})\, g(b_{n+1} \ldots b_1).$$ Now,

$$\begin{aligned}\| D_1(c)x_n \| \geq\ & \| D_1(a_n)\, b_n \ldots b_1\, x_n \| - \| \textstyle\sum_{k=1}^{n-1} D_1(a_k b_k \ldots b_1)\, x_n \| \\ & - \| a_n D_1(b_n \ldots b_1)\, x_n \| - \| g(a_n)\, h(b_n \ldots b_1)\, x_n \| \\ & - \| h(a_n)\, g(b_n \ldots b_1) x_n \| - \| D_1(c_{n+1})\, b_{n+1} \ldots b_1\, x_n \| \\ & - \| c_{n+1} D_1(b_{n+1} \ldots b_1) x_n \| - \| g(c_{n+1})\, h(b_{n+1} \ldots b_1)\, x_n \| \\ & - \| h(c_{n+1})\, g(b_{n+1} \ldots b_1)\, x_n \|,\end{aligned}$$ then by (1) we have

$$\begin{aligned}\| D_1(c)x_n \| \geq\ & n - \| a_n\, D_1(b_n \ldots b_1)\, x_n \| - \| g(a_n)\, h(b_n \ldots b_1)\, x_n \| \\ & - \| h(a_n)\, g(b_n \ldots b_1)\, x_n \| - \| c_{n+1}\, D_1(b_{n+1} \ldots b_1)\, x_n \| \\ & - \| g(c_{n+1}) h(b_{n+1} \ldots b_1) x_n \| - \| h(c_{n+1}) g(b_{n+1} \ldots b_1) x_n \| \quad ..(2)\end{aligned}$$

As a consequence, $$\| a_n\, D_1(b_n \ldots b_1)\, x_n \| \leq \| a_n \|\ \| D_1(b_n \ldots b_1) \| \leq 1 \quad \ldots\ldots (3)$$

Also, $$\| g(a_n) h(b_n \ldots b_1) x_n \| \leq \| g \|\, \| a_n \|\, \| h \|\, \| b_n \| \ldots \| b_1 \|\, \| x_n \| \leq \| a_n \| \leq 1 \quad \ldots\ldots (4)$$

Hence, $$\| h(a_n)\, g(b_n \ldots b_1) x_n \| \leq \| h \|\, \| a_n \|\, \| g \|\, \| b_n \| \ldots \| b_1 \|\, \| x_n \| \leq \| a_n \| \leq 1 \quad \ldots\ldots (5)$$

Now, we will follow the same way of [4] and [9], then we have

$\| c_{n+1} \| \leq 2 \| a_{n+1} \|$ ……. … (6)

So, $\| c_{n+1} D_1(b_{n+1} \dots b_1)x_n \| \leq \| c_{n+1} \| \| D_1(b_{n+1} \dots b_1) \|$, then by (6)

$\leq 2 \| a_{n+1} \| \| D_1(b_{n+1} \dots b_1) \|$

≤ 2 ……. … (7)

Also, $\| g(c_{n+1}) h(b_{n+1} \dots b_1) \; x_n \| \leq \| g \| \; \| c_{n+1} \| \; \| h \| \; \| b_{n+1} \| \dots$

$\| b_1 \| \; \| x_n \|$, then by (6)

$\leq 2 \| a_{n+1} \|$

≤ 2 ……. … (8)

And, $\| h(c_{n+1}) \; g(b_{n+1} \dots b_1) \; x_n \| \leq \| h \| \; \| c_{n+1} \| \; \| g \| \; \| b_{n+1} \| \dots$

$\| b_1 \| \| x_n \|$, then by (6)

$\leq 2 \| a_{n+1} \|$

≤ 2 ……. … (9)

Then by putting (3) , (4), (5), (7), (8) and (9) in (2) we get that $\| D_1(c)x_n \| \geq n - 9 \quad \forall\, n \in \mathbb{N}$, then $\| D_1(c) \| \geq \| D_1(c)x_n \| \geq n - 9$ $\forall\, n \in \mathbb{N}$. This contradiction proves our claim.

Let $m \in \mathbb{N}$ such that map $a \longmapsto D_1(a)x$ from I into X_{P_m} is continuous for some nonzero $x \in X_{P_m}$ and let X be the set of all $x \in X_{P_m}$ satisfying this property, X is a nonzero I– submodule of X_{P_m}; therefore, we conclude that $X = X_{P_m}$. Let $a \in S(D_1)$ then $\lim\limits_{n\to\infty} D_1(a_n) = a$ for a suitable sequence $\{a_n\}$ in I with $\lim\limits_{n\to\infty} a_n = 0$, then $ax = \lim\limits_{n\to\infty} D_1(a_n)x = 0$, for every $x \in X_{P_m}$ and therefore, $a \in P_m$. That means $S(D_1) \subset P_m$.

(ii) The proof is similar to the proof of that of first part of this proposition. ■

Proposition 3 : [9]

Let $P \in \mathcal{P}$ and J any subspace of $\mathcal{A}$ satisfying $IJ + JI \subset J$ and $J \not\subset P$. Then $J\,x = X_P$ for every nonzero $x \in X_P$.

Proof : see [9 , lemma 3]

Proposition 4 :

Let $P \in \mathcal{P}$ and J any non necessarily closed ideal of $\mathcal{A}$ contained in I. If there is an element $b \in J$ such that $b \notin P$, and $\dim bJb < \infty$. Then $S(D_1) \subset P$ and $S(D_2) \subset P$.

proof :

Note that, since $dim\ bJb < \infty$ then the map $a \longmapsto D(bJb)$ is continuous, let $a \in S(D)$, then there exists a sequence $\{a_n\} \subset I$ such that $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} D(a_n) = a$. Thus $\lim_{n\to\infty} b\, a_n b = 0$ and $\lim_{n\to\infty} D(ba_n b) = 0$. Since g and h are continuous linear maps, then $\lim_{n\to\infty} g(a_n) = 0$ and $\lim_{n\to\infty} h(a_n) = 0$, also $\lim_{n\to\infty} b\, a_n = 0$ thus $\lim_{n\to\infty} g(b\, a_n) = 0$ and $\lim_{n\to\infty} h(b\, a_n) = 0$.

Firstly, we will prove $S(D_1) \subset P$. Now, for all $b \in I, \{a_n\} \subset I$, we have:

$$\lim_{n\to\infty} D_1(b\, a_n\, b) = \lim_{n\to\infty} [D_1(ba_n)b + ba_n D_1(b) + g(ba_n)h(b) + h(ba_n)g(b)]$$
$$= \lim_{n\to\infty} [\, D_1(b)\, a_n\, b + b\, D_1(a_n)\, b + g(b)\, h(a_n)\, b + h(b)\, g(a_n)\, b + b\, a_n\, D_1(b) + g(b\, a_n)\, h(b) + h(b\, a_n)\, g(b)\,]$$
$$= b\, a\, b = 0 \quad \forall\, a \in S(D_1) \text{ hence } b\ S(D_1)\, b = 0$$

Since $b \notin P$ then $b\ X_P \neq 0$, if we assume that $S(D_1) \not\subset P$ then by Proposition 3 we have $S(D_1)\, b\ X_P = X_P$ thus $b\, S(D_1)\, b\ X_P = b\, X_P = 0$ Which gives $b \in P$ this is contradiction; therefore, $S(D_1) \subset P$.

Secondly, we will prove $S(D_2) \subset P$. Since $\lim_{n\to\infty} D_2(a_n) = a$; therefore, $\lim_{n\to\infty} b\ D_2(a_n) = b\ a$ this implies that $\lim_{n\to\infty} D_2(b\ a_n) = b\ a$, Now, for all $b \in I$, $\{a_n\} \subset I$, we have:

$$\lim_{n\to\infty} D_2(ba_n b) = \lim_{n\to\infty} [D_2(ba_n)b + ba_n D_1(b) + g(ba_n)h(b) + h(ba_n)\, g(b)]$$
$$= \lim_{n\to\infty} D_2(b\, a_n)\ b + \lim_{n\to\infty} b\, a_n\, D_1(b) + \lim_{n\to\infty} g(ba_n)\, h(b) + \lim_{n\to\infty} h(b\, a_n)\ g(b)$$
$$= b\, a\, b = 0 \ \ \forall\, a \in S(D_2) \text{ hence } b\ S(D_2)\ b = 0$$

Since $b \notin P$ then $b\ X_P \neq 0$, if we assume that $S(D_2) \not\subset P$ then by Proposition 3 we have $S(D_2)\, b\ X_P = X_P$ then $b\, S(D_2)\, b\ X_P = b\, X_P = 0$ that means $b \in P$ this is contradiction; therefore, $S(D_2) \subset P$. ■

The proof of the following result may be obtained in the same way as in [9 , theorem 5] applying the above propositions 2 and 4.

Proposition 5 : *D_1 and D_2 are closable.*

Proof : Obvious.

A Banach algebra $\mathcal{A}$ is said to be ultraprime if there exists a positive constant $K \geq 0$ such that $K \parallel a \parallel \parallel b \parallel \leq \parallel M_{a,b} \parallel \quad \forall a, b \in \mathcal{A}$, where $M_{a,b}$ is the tow - sided multipliplication operator on $\mathcal{A}$ defined by:
$M_{a,b}(x) = axb$ (see [9]).

In [3, proposition 2.3] it was proved that every prime C^* - algebra is an ultraprime Banach algebra, where $K = 1$.

Theorem 6 :
Let D_1 and D_2 be closable (g,h) -c- double derivation and generalized (g ,h) - c - double derivation respectively defined on a nonzero ideal I of an ultraprime Banach algebra, then D_1 and D_2 are continuous.

proof :
Since g and h are continuous; therefore, there are positive constants $\varepsilon, \delta \geq 0$ such that $\parallel g(y) \parallel \leq \varepsilon \parallel y \parallel$ and $\parallel h(z) \parallel \leq \delta \parallel z \parallel \quad \forall y, z \in \mathcal{A}$.

Firstly, we will prove D_1 is continuous. Fix $a \in I$, with $\parallel a \parallel = 1$ and consider the following mapping $f_1: \mathcal{A} \rightarrow \mathcal{A}$ define by $f_1(x) = D_1(xa)$ $\forall x \in \mathcal{A}$, we will follow the same way of [4] and [9], then we have f_1 is continuous; therefore, there is a positive constant $t \geq 0$, such that $\parallel f_1(x) \parallel \leq t \parallel x \parallel \quad \forall x \in \mathcal{A}$. Let $\parallel x \parallel = 1$ we have $\parallel f_1(x) \parallel \leq t$, thus $\parallel f_1(x) \parallel = \parallel D_1(xa) \parallel \leq t$. Now, for $b \in I$, $x \in \mathcal{A}$ we have : $D_1(bxa) = D_1(b)\, xa + b\, D_1(xa) + g(b)\, h(xa) + h(b)\, g(xa)$, then $D_1(b)\, x\, a = D_1(bxa) - b\, D_1(xa) - g(b)\, h(xa) - h(b)\, g(xa)$; therefore, $M_{D_1(b),a}(x) = D_1(bxa) - b\, D_1(xa) - g(b)\, h(xa) - h(b)\, g(xa)$, thus

$$\begin{aligned}\parallel M_{D_1(b),a}(x) \parallel &\leq \parallel D_1(bxa) \parallel + \parallel b\, D_1(xa) \parallel + \parallel g(b)\, h(xa) \parallel \\ &\quad + \parallel h(b)\, g(xa) \parallel \\ &\leq t + \parallel b \parallel t + \varepsilon \parallel b \parallel \delta \parallel xa \parallel + \delta \parallel b \parallel \varepsilon \parallel xa \parallel \\ &\leq 4\, t\, \varepsilon\, \delta \parallel b \parallel \parallel a \parallel.\end{aligned}$$

By taking supremum for both sides we have $\parallel M_{D_1(b),a} \parallel \leq 4t\varepsilon\delta \parallel b \parallel \parallel a \parallel$. Since $\mathcal{A}$ is ultraprime Banach algebra, then there exists a positive constant

$K \geq 0$ such that $K \| a \| \| b \| \leq \| M_{a,b} \|$, for all $a, b \in \mathcal{A}$. Then $K \| D_1(b) \| \| a \| \leq \| M_{D_1(b),a} \| \leq 4\, t\, \varepsilon\, \delta \| b \| \| a \|$, hence $\| D_1(b) \| \leq \frac{4\, t\, \varepsilon\, \delta}{K} \| b \|$, $\forall\, b \in I$. This implies that D_1 is continuous.

Secondly, we will prove D_2 is continuous. Fix $a \in I$, with $\| a \| = 1$ and consider the following mapping $f_2 : \mathcal{A} \to \mathcal{A}$ define by:

$$f_2(x) = D_2(x\, a)\ \forall x \in \mathcal{A},$$

we will follow the same way of [4] and [9], then we have f_2 is continuous; therefore, there is a positive constant $r \geq 0$, such that $\| f_2(x) \| \leq r \| x \|\ \ \forall\, x \in \mathcal{A}$. Let $\| x \| = 1$ we have $\| f_2(x) \| \leq r$, thus $\| f_2(x) \| = \| D_2(xa) \| \leq r$. Now, for $b \in I$, $x \in \mathcal{A}$ we have : $D_2(b\, x\, a) = D_2(b)\ x\ a + b\, D_1(xa) + g(b)\ h(xa) + h(b)\, g(xa)$, so $D_2(b)\ x\ a = D_2(bxa) - bD_1(xa) - g(b)h(xa) - h(b)g(xa)$; therefore, $M_{D_2(b),a}(x) = D_2(bxa) - b\ D_1(xa) - g(b)\, h(xa) - h(b)\, g(xa)$, thus

$$\begin{aligned}\| M_{D_2(b),a}(x) \| &\leq \| D_2\,(bxa) \| + \| b\ D_1(x\,a) \| + \| g(b)\ h(x\,a) \| \\ &\quad + \| h(b)\, g(xa) \| \\ &\leq r + \| b \| \frac{4t\varepsilon\delta}{K} \| xa \| + \varepsilon \| b \|\, \delta \| xa \| + \delta \| b \|\, \varepsilon \| xa \| \\ &\leq 7\, r\, t\, \varepsilon\, \delta \| b \| \| a \|.\end{aligned}$$

By taking supremum for both sides we get $\| M_{D_2(b),a} \| \leq 7rt\varepsilon\delta \| b \| \| a \|$. Since $\mathcal{A}$ is ultraprime Banach algebra, then there exists a positive constant $m \geq 0$ such that $m \| a \| \| b \| \leq \| M_{a,b} \|$, for all $a, b \in \mathcal{A}$. Then $m \| D_2(b) \| \| a \| \leq \| M_{D_2(b),a} \| \leq 7\, r\, t\, \varepsilon\, \delta \| b \| \| a \|$, hence $\| D_2(b) \| \leq \frac{7\, rt\varepsilon\delta}{m} \| b \|$, $\forall\, b \in I$. This proves that D_2 is continuous. ∎

Applying proposition 5 and theorem 6 we can prove the following result :

Corollary 7 :

Every essentially defined (g,h) - c -double derivation and generalized (g , h) - c - double derivation on a nonzero ideal of prime C^* - algebra is continuous.

Corollary 8 :

Every essentially defined derivation on a nonzero ideal of prime C^* - algebra is continuous.

Proof:

i) By corollary 7, taking g or h or both in D_1 to be the zero maps.

ii) By corollary 7, let $D_1 = D_2$ and taking g or h or both in D_2 to be the zero maps.

Remark 9 :

The above results of this paper are also true for the following derivations:

(1) *$D_3 : I \to \mathcal{A}$ such that $D_3(ab)= D_3(a)\, g(b) + h(a)\, D_3(b)$, for all $a,b \in I$.*

(2) *$D_4 : I \to \mathcal{A}$ such that $D_4(ab)= D_4(a)\, g(b) + h(a)\, D_3(b)$, for all $a,b \in I$.*

References

[1] Johnson B. E. and Sinclair A. M. , " Continuity of derivations and a problem of Kaplansky " , Amer. J. Math. , 90 : 1067 - 1073 (1968) .

[2] Lee T – K . and Liu C – K . , " Partially defined σ – derivations on semisimple Banach algebras" , Studia. Math. , 190 : 193 - 202 (2009).

[3] Mathieu M. , " Elementary operator on prime $C*$ - algebra " , Math. Ann. , 284 : 223 - 244 (1989) .

[4] Mohammed A. A. and Ali S. M. , " On Villena's theorem of automatic continuity of essentially defined derivations on semisimple Banach algebras " , Int. J. of Math. Analysis, 7 : 2931 - 2939 (2013).

[5] Mahdavian Rad H. and Niknam A. , " Double derivations, higher double derivations and automatic continuity " , J. of sciences, 24 (2) : 165 - 170 (2013) .

[6] Mirzavaziri M. and Moslehian M. S. , " Automatic continuity of σ – derivations on $C*$ - algebras " , Proc. Amer. Math. Soc. , 134 : 3319 - 3327 (2006) .

[7] Mirzavaziri M. and Omidvar Tehrani E. , " (δ , ϵ) - double derivations on $C*$ - algebras " , Bull. Iranian Math. Soc. , 35 : 147 - 154 (2009).

[8] Parky C. and Yun Shinz D. , " Generalized (θ , ø) - derivations on Banach algebras ", Korean J. Math. , 22 (1) : 139 - 150 (2014) .

[9] Villena A. R. , " Essentially defined derivations on semisimple Banach algebras " , Proc. Edinburgh Math. Soc. , 40 : 175 - 179 (1997) .