On Solutions of a Quadratic Integral Equation of Urysohn Type

Lamyaa H. Sadoon
Department of Mathematics / College of Education
University of Mosul

Received 29 / 12 / 2011 **Accepted** 02 / 05 / 2012

الملخص

في هذا البحث ندرس وجود حل وحيد على فترة شبه لانهائية لمعادلة تكاملية ثنائية من نوع يوريسون في فضاء فرشت باستخدام نوع متناوب غير خطي لشوادر- لاري للتقابلات الانكماشية.

Abstract

In this paper, we investigate the existence of a unique solution on a semi-infinite interval for a quadratic integral equation of Urysohn type in Frechet space using a nonlinear alternative of Leray-Schauder type for contraction maps.

***Keywords*:- quadratic integral equation, existence and uniqueness, fixed point, Leray-Schauder, Frechet space.**

1. Introduction

In this paper, we establish the existence of the unique solution, defined on a semi-finite interval $J=[0,+\infty)$ for a quadratic integral equation of Urysohn type, namely

$$x(t)= f(t)+(Ax)(t)\left(\int_0^T u(t,s,x(s))ds+\int_0^t g(t,s,x(s))ds\right),\quad t\in J:=[0,+\infty)$$

where $f:J\to R$, $u:J\times[0,T]\times R\to R$ and $g:J\times[0,T]\times R\to R$ are given functions and $A:c(J,R)\to c(J,R)$ is an appropriate operator, here $c(J,R)$ denotes the space of continuous functions $x:J\to R$.

Integral equations occur naturally in many fields of mechanics and mathematical physics. They also arise as representation formulas for the solutions of differential equations. Indeed, a differential equation can be replaced by an integral equation which incorporates its boundary conditions. As such, each solution of the integral equation automatically satisfies these boundary conditions.

The theory of Volterra –Fredholm integral equations play an important role for abstract formulation of many initial, boundary value problems of perturbed differential equations, partial differential equations and partial integro differential equations which arise in various applications like chemical reaction kinetics, population dynamics, heat flow in material with memory, viscoelastic and reaction diffusion problems. Instance, we refer to [1,3,6,7,8].

Quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory, the quadratic integral equation can be very often encountered in many applications see [4,5].

Recently, the existence of a unique solution for the nonlinear quadratic integral equation of Urysohn type

$$x(t) = f(t) + (Ax)(t)\int_0^T u(t,s,x(s))ds \quad , \quad t \in [0,+\infty)$$

was studied in [2] by using a nonlinear alternative of Leray-Schauder type.

Here we are concerning with the nonlinear quadratic integral equation of Fredholm – Volterra integral equation

$$x(t) = f(t) + (Ax)(t)\left(\int_0^T u(t,s,x(s))ds + \int_0^t g(t,s,x(s))ds\right) \quad , \quad t \in [0,+\infty) \qquad (1.1)$$

By using the same assumptions assumed in [2].

2. preliminaries

We introduce some notations, definitions and theorems which are used throughout this paper.

Let X be Frechet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in N}$. Let $Y \subset X$, we say that Y is bounded if for every $n \in N$, these exists $M_n > 0$ such that

$$\|y\|_n \le M_n \text{ for all } y \in Y.$$

Definition 2.1 [2]: A function $f : X \to X$ is said to be a contraction mapping if for each $n \in N$ there exists $k_n \in (0,1)$ such that:

$$\| f(x) - f(y)\|_n \le k_n \|x - y\|_n \qquad \text{for all } x, y \in X.$$

Theorem 2.2 [2] : Let Ω be a closed subset of a Frechet space X such that $0 \in \Omega$ and $F : \Omega \to X$ is a contraction such that $F(\Omega)$ is bounded. Then either

1) F has a unique fixed point or
2) there

exists $\lambda \in (0,1), n \in N$ *and* $u \in \partial\Omega^n$ *such that* $\|u - \lambda F(u)\|_n = 0$.

Where $\partial\Omega^n$ is boundary of Ω^n.

3. Main theorem

In this section, we assume that the following assumptions are satisfied:

i) $f : J \to R$ is a continuous function.

ii) For each $n \in N$ $\exists$ $L_n > 0$ $s.t.$ $|(Ax)(t) - (A\bar{x})(t)| \le L_n |x(t) - \bar{x}(t)|$ for each $x, \bar{x} \in c(J,R)$ and $t \in [0,n]$.

iii) There exists nonnegative constants a, b such that:
$|(Ax)(t)| \le a + b|x(t)|$ $for\ each$ $x \in c(J,R)$ and $t \in J$.

iv) $u : J \times J \times R \to R$ is continuous function and for each $n \in N$ there exist a constant $L_n^* > 0$ such that:
$|u(t,s,x) - u(t,s,\bar{x})| \le L_n^* |x - \bar{x}|$ $for\ all$ $(t,s) \in [0,T]$ and $x, \bar{x} \in R$ and
$g : J \times J \times R \to R$ is continuous function and for each $n \in N$ there exist a constant $H_n^* > 0$ such that:
$|g(t,s,x) - g(t,s,\bar{x})| \le H_n^* |x - \bar{x}|$ $for\ all$ $(t,s) \in [0,T]$ and $x, \bar{x} \in R$.

v) There exist a continuous non decreasing function $\psi : J \to (0,\infty)$ and
$p \in c(J, R_+)$ $such\ that$ $|u(t,s,x)| \le p(s)\psi(|x|)$ $for\ each$ $(t,s) \in J \times [0,T]$, $x \in R$
and $\varphi : J \to (0,\infty)$ with:
$q \in c(J, R_+)$ $such\ that$ $|g(t,s,x)| \le q(s)\varphi(|x|)$ $for\ each$ $(t,s) \in J \times [0,T]$, $x \in R$
and moreover there exists a constants $M_n, n \in N$ such that:

$$\frac{M_n}{\|f\|_n + (a + bM_n)T(\psi(M_n)p^* + \varphi(M_n)q^*)} > 1 \tag{3.1}$$

Where $p^* = \sup\{p(s) : s \in [0,T]\}$ and $q^* = \sup\{q(s) : s \in [0,T]\}$

Theorem 3.1

Suppose that hypotheses (i-v) are satisfied. If

$$(a + bM_n)T(L_n^* + H_n^*) + TL_n(\psi(M_n)p^* + \varphi(M_n)q^*) < 1 \tag{3.2}$$

Then equation (1.1) has a unique solution.

Proof : For every $n \in N$, we define in $c(J,R)$ the semi-norms by $\|y\|_n = \sup\{|y(t)| : t \in [0,n]\}$ then $c(J,R)$ is a Frechet space with the family of semi-norms $\{\|.\|\}_{n \in N}$ [2].

Transform the problem (1.1) into a fixed point problem. Consider the operator
$F : c(J,R) \to c(J,R)$ defined by

$$(Fy)(t) = f(t) + (Ay)(t)\left(\int_0^T u(t,s,y(s))ds + \int_0^t g(t,s,y(s))ds\right), t \in J$$

Let y be a possible solution of the problem (1.1). Given $n \in N$ and $t \le n$, then with the view of (i),(iii) and (v) we have:

$$|\ y(t)\ | \le |\ f(t)\ | + |\ (Ay)(t)\ | (\int_0^T |\ u(t,s,y(s))\ | ds + \int_0^t |\ g(t,s,y(s))\ | ds\ ),$$

$$\le |f(t)| + (a + b|y(t)|)(\int_0^T p(s)\psi(|y(s)|)ds + \int_0^t q(s)\varphi(|y(s)|)ds)$$

$$\|y\|_n \le \|f\|_n + (a + b(\|y\|_n)T(\psi(\|y\|_n)p^* + \varphi(\|y\|_n)q^*)$$

Then

$$\frac{\|y\|_n}{\|f\|_n + (a + b\|y\|_n)T(\psi(\|y\|_n)p^* + \varphi(\|y\|_n)q^*)} \le 1$$

From (3.1) it follows that for each $n \in N$, $\|y\|_n \ne M_n$

Now, set

$\Omega = \{y \in c(J,R) : \|y\|_n \le M_n \quad for\ all\ \ n \in N\}$

Clearly, Ω is a closed subset of $c(J,R)$, we shall show that $F:\Omega \to c(J,R)$ is a contraction operator, consider $y, \bar{y} \in \Omega$ *for each* $t \in [0,n]$ *and* $n \in N$ from (ii)-(iv) we have:

$$|(Fy)(t) - (F\bar{y})(t)| \le |(Ay)(t)(\int_0^T |\ u(t,s,y(s))ds + \int_0^t g(t,s,y(s))ds) -$$

$$-(A\bar{y})(t)(\int_0^T u(t,s,\bar{y}(s))ds + \int_0^t g(t,s,\bar{y}(s))ds)\ |$$

$$\le |(Ay)(t)(\int_0^T |\ u(t,s,y(s))ds + \int_0^t g(t,s,y(s))ds) -$$

$$-(Ay)(t)(\int_0^T u(t,s,\bar{y}(s))ds + \int_0^t g(t,s,\bar{y}(s))ds)\ | +$$

$$+|\ (Ay)(t)(\int_0^T u(t,s,\bar{y}(s))ds + \int_0^t g(t,s,\bar{y}(s))ds) -$$

$$-(A\bar{y})(t)(\int_0^T u(t,s,\bar{y}(s))ds + \int_0^t g(t,s,\bar{y}(s))ds)\ |$$

$$\le |(Ay)(t)(\int_0^T |\ u(t,s,y(s)) - u(t,s,\bar{y}(s))| ds + \int_0^t\ |g(t,s,y(s)) -$$

$$-g(t,s,\bar{y}(s))|ds) + |(Ay)(t) - (A\bar{y})(t)\ |(\int_0^T |\ u(t,s,\bar{y}(s))\ | ds +$$

$$+\int_0^t |\ g(t,s,\bar{y}(s))\ | ds)$$

$$\le (a+b(|y(t)|)(L_n^*\int_0^T |(y(s)-\bar{y}(s)|ds + H_n^*\int_0^t |(y(s)-\bar{y}(s)|ds)+$$

$$+L_n|(y(t)-\bar{y}(t)|(\int_0^T p(s)\psi(|(\bar{y}(s)|ds + \int_0^t q(s)\varphi(|(\bar{y}(s)|ds)$$

$$\le \|y-\bar{y}\|_n[(a+bM_n)T(L_n^*+H_n^*)+TL_n(\psi(M_n)p^*+\varphi(M_n)q^*)]$$

Therefore

$$\|Fy-F\bar{y}\| \le \|y-\bar{y}\|_n[T(a+bM_n)(L_n^*+H_n^*)+TL_n(\psi(M_n)p^*+\varphi(M_n)q^*]$$

By (3.2) F is a contraction for all $n \in N$. From the choice of Ω is no $y \in \partial\Omega$ such that $y = \lambda F(y)$ for some $\lambda \in (0,1)$. Then the statement (2) in the theorem (2.2) does not hold. The nonlinear alternative of Leray-Schauder type [2] shows that (1) holds, and hence we deduced that the operator F has a unique fixed point y in Ω which is a solution of equation (1.1).

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