The existence and approximation of the periodic solutions for system of first order nonlinear differential equationsby using Lebesgue integrable

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Received 08/10/2006
Accepted 09/01/2007

الملخص

يتضمن هذا البحث دراسة وجود وتقارب الحـل الـدوري لنظـام مـن المعـادلات التفاضليـة اللاخطيـة مـن الرتبـة الاولـى وبـافتراض أن كـل مـن الدالتيـن $g(t,x,y), f(t,x,y)$ قابلتان للقياس عند t ومقيدتين بدالتين قابلتان للتكامـل الليبيكي وذلك باستخدام الطريقة التحليليـة ـ العددية لدراسة الحلول الدورية للمعادلات التفاضلية الاعتياديـة اللاخطية لـ ـ Samoilenko.

ABSTRACT

In this paper we study the existence and approximation of the periodic solutions for a system of first order nonlinear differential equations by assuming that each of the functions $f(t,x,y), g(t,x,y)$ are measurable at t and bounded by Lebesgue integrable functions.

The numerical-analytic method has been used to study the periodic solutions of ordinary differential equations which were introduced by A. M. Samoilenko.

INTRODUCTION

There are many subjects in physics and technology use mathematical methods that depends on the nonlinear differential equations, and it became clear that the existence of the periodic solutions and its algorithm structure form an important problems in the present time, where many of studies treated autonomous and non autonomous periodic systems and specially with the integral and differential equations and linear and nonlinear integro - differential problems of periodic solutions.

Samoilenko [7] assumed a numerical analytic method to study the periodic solutions for the ordinary differential equations and this method include uniformly sequences of the periodic functions as in the studies [1,2,3,5,6].

In this paper we consider the system of the non linear differential equations of the form:

$$\left.\begin{aligned}\frac{dx(t)}{dt} &= (A+B(t))x(t)+f(t,x,y)\\ \frac{dy(t)}{dt} &= (C+D(t))y(t)+g(t,x,y)\end{aligned}\right\} \qquad \text{........(1)}$$

where $x \in D \subseteq R^n, D$ represents a closed domain and bounded , the each of the functions

$$f(t,x,y) = (f_1(t,x,y), f_2(t,x,y), \ldots\ldots, f_n(t,x,y))$$

$$g(t,x,y) = (g_1(t,x,y), g_2(t,x,y), \ldots\ldots, g_n(t,x,y))$$

are defined, and continuous in the domain

$$(t,x,y) \in [0,T] \times D \times D_1 \qquad \text{........(2)}$$

and periodic in t of period T.

where $y \in D_1 \subseteq R^n, D_1$ represents a closed domain and bounded from the Euclidean space R^n, with the assumption that the two functions $f(t,x,y)$ and $g(t,x,y)$ satisfy the following inequalities:

$$\|f(t,x,y)\| \le m_1(t) \quad , \quad \|g(t,x,y)\| \le m_2(t) \qquad \text{.......(3)}$$

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le K_1(t)\|x_1 - x_2\| + K_2(t)\|y_1 - y_2\| \qquad \text{.......(4)}$$

$$\|g(t,x_1,y_1) - g(t,x_2,y_{22})\| \le L_1(t)\|x_1 - x_2\| + L_2(t)\|y_1 - y_2\| \qquad \text{.......(5)}$$

for all $t \in [0,T], x, x_1, x_2 \in D, y, y_1, y_2 \in D_1$ where each of $m_1(t), m_2(t)$, $K_1(t), K_2(t), L_1(t), L_2(t)$ are Lebesgue integrable functions in the interval $0 \le t \le T$, where $A=[A_{ij}], B(t)=[B_{ij}(t)], C=[C_{ij}], D(t)=[D_{ij}(t)]$ positive n×n matrices are defined in the domain $-\infty, < 0 \le t \le T < \infty$ continuous and periodic in t and satisfy the following inequalities :

$$\|e^{A(t-s)}\| \le Q \quad , \quad \|e^{C(t-s)}\| \le R \qquad \text{..........(6)}$$

$$\|B(t)\| \le H \quad , \quad \|D(t)\| \le J \qquad \text{..........(7)}$$

$$\|x_\circ\| = \delta_\circ \quad , \quad \|y_\circ\| = \sigma_\circ \qquad \text{...........(8)}$$

Where $-\infty, < 0 \le t \le T < \infty$ and Q , R , H , J, $\delta_\circ, \sigma_\circ$ are positive constants.

Let $0 \le t \le b_1 \le T$ *and* $0 \le t \le b_2 \le T$ where b_1, b_2 are two chosen points so that

$$\int_0^{b_1} m_1(t)\,dt \le c_1 \ , \ \int_0^{b_1} K_1(t)\,dt \le c_2 \ \le 1 \ , \ \int_0^{b_1} K_2(t)\,dt \le c_3 \ \le 1 \qquad \text{........(9)}$$

$$\int_0^{b_2} m_2(t)\,dt \le \delta_1 \ , \ \int_0^{b_2} L_1(t)\,dt \le \delta_2 \ \le 1 \ , \ \int_0^{b_2} L_2(t)\,dt \le \delta_3 \ \le 1 \qquad ....(10)$$

Definition 1 [7] :-

The system of nonlinear differential equations (1) where the right-hand side is defined, continuous and periodic in t and has period T in the domain (2) is said to be system – T if

1- The two sets D_f, D_g are not empty

$$\left.\begin{aligned} D_f &= D - (\frac{T}{2}N + QC_1) \ne \phi \\ D_g &= D_1 - (\frac{T}{2}N^* + R\delta_1) \ne \phi \end{aligned}\right\} \qquad ....(11)$$

where $N^* = R^2 J\sigma_\circ, N = Q^2 H\delta_\circ$, $\|.\| = \max_{t\in[0.T]}|.|$

2- The greatest eigen value of the matrix

$$\Omega_\circ = \begin{bmatrix} \frac{T}{2}QH + QC_2 & QC_3 \\ R\delta_2 & \frac{T}{2}RJ + R\delta_3 \end{bmatrix}$$

is smaller than 1 that is

$$\lambda_{\max}(\Omega_\circ) = \frac{\rho_1 + \sqrt{\rho_1^2 - 4\rho_2}}{2} < 1 \qquad ......(12)$$

Where

$$\rho_1 = (\frac{T}{2}QH + QC_2) + (\frac{T}{2}RJ + R\delta_3)$$

$$\rho_2 = R\delta_2 QC_3 - (\frac{T}{2}QH + QC_2)(\frac{T}{2}RJ + R\delta_3)$$

Definition 2 [7]:-

The value of $\mu^* = (\mu_1^*, \mu_2^*)$ at the point $(t, x_\circ, y_\circ)$ for the following system:

$$\left.\begin{aligned} \frac{dx(t)}{dt} &= (A + B(t))x(t) + f(t,x,y) - \mu_1^* \\ \frac{dy(t)}{dt} &= (C + D(t))y(t) + g(t,x,y) - \mu_2^* \end{aligned}\right\} \qquad ....(13)$$

Is periodic in t of period T is called a constant - Δ^* for the system (1) through at the point $t = 0, x = x_\circ, y = y_\circ$ if μ^* is unique at that point.

Section One : The periodic approximate solution for the system (1)

Lemma 1 :-

Assume that each of $f(t,x,y)$ *and* $g(t,x,y)$ are vector functions, continuous and defined in the interval [0,T], then the inequality

$$\begin{pmatrix} \|M_1(t,x_\circ,y_\circ)\| \\ \|M_2(t,x_\circ,y_\circ)\| \end{pmatrix} \le \begin{pmatrix} \beta_1(t)N + QC_1 \\ \beta_2(t)N^* + R\delta_1 \end{pmatrix} \qquad \text{........(1.1)}$$

holds for $0 \le t \le T$, $\beta_1(t) \le \frac{T}{2}$, $\beta_2(t) \le \frac{T}{2}$, where

$N^* = R^2 J\sigma_\circ , N = Q^2 H\delta_\circ$

$$\beta_1(t) = \left[\frac{t(2e^{\|A\|(T-t)} - e^{\|A\|T} - \|E\|) + T\,(e^{\|A\|T} - e^{\|A\|(T-t)})}{e^{\|A\|T} - \|E\|} \right]$$

$$\beta_2(t) = \left[\frac{t(2e^{\|C\|(T-t)} - e^{\|C\|T} - \|E\|) + T\,(e^{\|C\|T} - e^{\|C\|(T-t)})}{e^{\|C\|T} - \|E\|} \right] ,$$

$$M_1(t,x_\circ,y_\circ) = \int_0^t e^{A(t-s)} \big[B(s)x_\circ e^{At} + f(s,x_\circ,y_\circ) -$$

$$- \frac{A}{e^{AT} - E} \int_0^T e^{A(T-s)} [B(s)x_\circ e^{At} + f(s,x_\circ,y_\circ)]ds\big]ds$$

$$M_2(t,x_\circ,y_\circ) = \int_0^t e^{C(t-s)} \big[D(s)y_\circ e^{Ct} + g(s,x_\circ,y_\circ) -$$

$$- \frac{C}{e^{CT} - E} \int_0^T e^{C(T-s)} [D(s)y_\circ e^{Ct} + g(s,x_\circ,y_\circ)]ds\big]ds$$

Proof :-

$\|M_1(t,x_\circ,y_\circ)\| \le$

$$\le \left[\|E\| - \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|} \right) \right] \int_0^t \|e^{A(t-s)}\| [\|B(s)\|\|x_\circ\|\|e^{At}\| + \|f(s,x_\circ,y_\circ)\|]ds +$$

$$+ \frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|} \int_t^T \|e^{A(t-s)}\| [\|B(s)\|\|x_\circ\|\|e^{At}\| + \|f(s,x_\circ,y_\circ)\|]ds$$

$$= \beta_1(t)N + QC_1 \qquad \text{........(1.2)}$$

and also

$$\|M_2(t,x_\circ,y_\circ)\| \le$$

$$\le \left[\|E\| - \left(\frac{e^{\|C\|T} - e^{\|C\|(T-t)}}{e^{\|C\|T} - \|E\|}\right)\right] \int_0^t \|e^{C(t-s)}\| [\,\|D(s)\|\,\|y_\circ\|\|e^{Ct}\| + \|g(s,x_\circ,y_\circ)\|] ds +$$

$$+ \frac{e^{\|C\|T} - e^{\|C\|(T-t)}}{e^{\|C\|T} - \|E\|} \int_t^T \|e^{C(t-s)}\| [\|D(s)\|\|y_\circ\|\|e^{Ct}\| + \|g(s,x_\circ,y_\circ)\|] ds$$

$$= \beta_2(t) N^* + R\delta_1 \qquad \text{........(1.3)}$$

from (1.2) and (1.3) we conclude that the inequality (1.1) holds for $0 \le t \le T$, $\beta_1(t) \le \frac{T}{2}$, $\beta_2(t) \le \frac{T}{2}$

Theorem 1:-

If the system (1) which satisfies the inequalities (3),(4),(5) and the conditions (11), (12) has a periodic solution $x = x(t,x_\circ,y_\circ)$, $y = y(t,x_\circ,y_\circ)$ passes through the point $(t,x_\circ,y_\circ)$, then the sequences of functions:-

$$x_{m+1}(t,x_\circ,y_\circ) = x_\circ e^{At} + \int_0^t e^{A(t-s)} [B(s)\, x_m(s,x_\circ,y_\circ) + f(s, x_m(s,x_\circ,y_\circ), y_m(s,x_\circ,y_\circ)) -$$

$$- \frac{A}{e^{AT} - E} \int_0^T e^{A(T-s)} [B(s)\, x_m(s,x_\circ,y_\circ) + f(s, x_m(s,x_\circ,y_\circ), y_m(s,x_\circ,y_\circ))] ds] ds$$

........(1.4)

with $x_\circ(t,x_\circ,y_\circ) = x_\circ e^{At}$ $, m = 0,1,2,.....$

$$y_{m+1}(t,x_\circ,y_\circ) = y_\circ e^{Ct} + \int_0^t e^{C(t-s)} [D(s)\, y_m(s,x_\circ,y_\circ) + g(s, x_m(s,x_\circ,y_\circ), y_m(s,x_\circ,y_\circ)) -$$

$$- \frac{C}{e^{CT} - E} \int_0^T e^{C(T-s)} [D(s)\, y_m(s,x_\circ,y_\circ) + g(s, x_m(s,x_\circ,y_\circ), y_m(s,x_\circ,y_\circ))] ds] ds$$

........(1.5)

with $y_\circ(t,x_\circ,y_\circ) = y_\circ e^{Ct}$ $, m = 0,1,2,........$

are periodic in t of period T and uniformly convergent as $m \to \infty$ in the domain

$$(t,x_\circ,y_\circ) \in [0,T] \times D_f \times D_g \qquad \text{............(1.6)}$$

to the limit functions $x^{\circ}(t,x_{\circ},y_{\circ})$ *and* $y^{\circ}(t,x_{\circ},y_{\circ})$ which are defined, continuous and periodic in t of period T in the domain (1.6) satisfy the system of integral equations

$$x(t,x_{\circ},y_{\circ})=x_{\circ}e^{At}+\int_0^t e^{A(t-s)}[B(s)\,x(s,x_{\circ},y_{\circ})+f(s,x(s,x_{\circ},y_{\circ}),y(s,x_{\circ},y_{\circ}))-$$

$$-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)\,x(s,x_{\circ},y_{\circ})+f(s,x(s,x_{\circ},y_{\circ}),y(s,x_{\circ},y_{\circ}))]ds]ds$$

...........(1.7)

$$y(t,x_{\circ},y_{\circ})=y_{\circ}e^{Ct}+\int_0^t e^{C(t-s)}[D(s)\,y(s,x_{\circ},y_{\circ})+g(s,x(s,x_{\circ},y_{\circ}),y(s,x_{\circ},y_{\circ}))-$$

$$-\frac{C}{e^{CT}-E}\int_0^T e^{C(T-s)}[D(s)\,y(s,x_{\circ},y_{\circ})+g(s,x(s,x_{\circ},y_{\circ}),y(s,x_{\circ},y_{\circ}))]ds]ds$$

...........(1.8)

which are a unique solution of the system (1) provided that:

$$\begin{pmatrix} \|x^{\circ}(t,x_{\circ},y_{\circ})-x_m(t,x_{\circ},y_{\circ})\| \\ \|y^{\circ}(t,x_{\circ},y_{\circ})-y_m(t,x_{\circ},y_{\circ})\| \end{pmatrix} \le \Omega_{\circ}^{m}(E-\Omega_{\circ})^{-1}Z_{\circ} \qquad ....(1.9)$$

for all $m\ge 1$ and $t\in R'$.

where $Z_{\circ}=\begin{pmatrix} \frac{T}{2}N+QC_1 \\ \frac{T}{2}N^{*}+R\delta_1 \end{pmatrix}$, E the identity matrix

Proof :-

Each of the following sequences which are defined by (1.4) and (1.5) and has the form

$$x_1(t,x_{\circ},y_{\circ}),x_2(t,x_{\circ},y_{\circ}),......,x_m(t,x_{\circ},y_{\circ}),.......$$

$$y_1(t,x_{\circ},y_{\circ}),y_2(t,x_{\circ},y_{\circ}),......,y_m(t,x_{\circ},y_{\circ}),.......$$

are defined , continuous in the domain (2) and periodic in t of period T .

By lemma 1 and from (1.4) when m=0 we obtain

$$\|x_1(t,x_{\circ},y_{\circ})-x_{\circ}(t,x_{\circ},y_{\circ})\|\le\beta_1(t)\,N+QC_1\le\frac{T}{2}N+QC_1 \qquad ....(1.10)$$

Thus $x_1(t,x_{\circ},y_{\circ})\in D$ for all $x_{\circ}\in D_f\,,y_{\circ}\in D_g$

also from the relation (19) and by lemma 1 when m=0 we obtain

$$\|y_1(t,x_{\circ},y_{\circ})-y_{\circ}(t,x_{\circ},y_{\circ})\|\le\beta_2(t)N^{*}+R\delta_1\le\frac{T}{2}N^{*}+R\delta_1 \qquad ....(1.11)$$

and this gives $y_1(t,x_\circ,y_\circ)\in D_1$ for all $x_\circ\in D_f$, $y_\circ\in D_g$

By using the mathematical induction we can prove the truth of the following inequalities for $m\geq 1$

$$\left.\begin{aligned}\|x_m(t,x_\circ,y_\circ)-x_\circ(t,x_\circ,y_\circ)\|&\leq \frac{T}{2}N+QC_1\\ \|y_m(t,x_\circ,y_\circ)-y_\circ(t,x_\circ,y_\circ)\|&\leq \frac{T}{2}N^*+R\delta_1\end{aligned}\right\} \qquad ....(1.12)$$

that is $x_m(t,x_\circ,y_\circ)\in D, y_m(t,x_\circ,y_\circ)\in D_1$ for all $x_\circ\in D_f$, $y_\circ\in D_g$

Now we prove that the sequences of functions $\{x_m(t,x_\circ,y_\circ)\}_{m=0}^{\infty}$ and $\{y_m(t,x_\circ,y_\circ)\}_{m=0}^{\infty}$ are uniformly convergent in the domain (1.6) and also each the sequences of functions are periodic and continuous in the same domain.

Now when m=1 in (1.4) and by using lemma 1 we find that

$$\|x_2(t,x_\circ,y_\circ)-x_1(t,x_\circ,y_\circ)\|\leq[\beta_1(t)QH+QC_2]\|x_1(t,x_\circ,y_\circ)-x_\circ(t,x_\circ,y_\circ)\|+ \\ +QC_3\|y_1(t,x_\circ,y_\circ)-y_\circ(t,x_\circ,y_\circ)\|$$

Also when m=1 in (1.4) and by using lemma 1 we find that

$$\|y_2(t,x_\circ,y_\circ)-y_1(t,x_\circ,y_\circ)\|\leq R\delta_2\|x_1(t,x_\circ,y_\circ)-x_\circ(t,x_\circ,y_\circ)\|+ \\ +[\beta_2(t)RJ+R\delta_3]\|y_1(t,x_\circ,y_\circ)-y_\circ(t,x_\circ,y_\circ)\|$$

So by using the mathematical induction we can prove the truth of the following inequalities:

$$\|x_{m+1}(t,x_\circ,y_\circ)-x_m(t,x_\circ,y_\circ)\|\leq[\beta_1(t)QH+QC_2]\|x_m(t,x_\circ,y_\circ)-x_{m-1}(t,x_\circ,y_\circ)\|+ \\ +QC_3\|y_m(t,x_\circ,y_\circ)-y_{m-1}(t,x_\circ,y_\circ)\| \qquad ....(1.13)$$

$$\|y_{m+1}(t,x_\circ,y_\circ)-y_m(t,x_\circ,y_\circ)\|\leq R\delta_2\|x_m(t,x_\circ,y_\circ)-x_{m-1}(t,x_\circ,y_\circ)\|+ \\ +[\beta_2(t)RJ+R\delta_3]\|y_m(t,x_\circ,y_\circ)-y_{m-1}(t,x_\circ,y_\circ)\|$$

........(1.14)

we rewrite (1.13) and (1.14) in vector form as

$$Z_{m+1}(t)\leq\Omega(t)Z_m(t) \qquad ....(1.15)$$

where

$$Z_{m+1}(t)=\begin{pmatrix}\|x_{m+1}(t,x_\circ,y_\circ)-x_m(t,x_\circ,y_\circ)\|\\ \|y_{m+1}(t,x_\circ,y_\circ)-y_m(t,x_\circ,y_\circ)\|\end{pmatrix}$$

$$\Omega(t)=\begin{bmatrix}\beta_1(t)QH+QC_2 & QC_3\\ R\delta_2 & \beta_2(t)RJ+R\delta_3\end{bmatrix}$$

$$Z_m(t) = \begin{pmatrix} \|x_m(t,x_\circ,y_\circ) - x_{m-1}(t,x_\circ,y_\circ)\| \\ \|y_m(t,x_\circ,y_\circ) - y_{m-1}(t,x_\circ,y_\circ)\| \end{pmatrix}$$

Now we take the maximum value for the two sides of the inequality (1.15) with

$$\beta_2(t) \le \frac{T}{2} \;, \beta_1(t) \le \frac{T}{2} \;, 0 \le t \le T$$

we find that

$$Z_{m+1} \le \Omega_\circ Z_m \qquad \text{...........(1.16)}$$

where $\Omega_\circ = \max\limits_{t\in[0,T]} \Omega(t)$,

$$\Omega_\circ = \begin{bmatrix} \frac{T}{2}QH + QC_2 & QC_3 \\ R\delta_2 & \frac{T}{2}RJ + R\delta_3 \end{bmatrix}$$

and by (1.16) we have

$$Z_{m+1} \le \Omega_\circ^m Z_\circ \;,$$

and then

$$\sum_{i=1}^{m} Z_i \le \sum_{i=1}^{m} \Omega_\circ^{i-1} Z_\circ \,, \qquad \text{..........(1.17)}$$

since the matrix $\Omega_\circ$ has the greatest eigen value given by (12), this insures that the sequence (1.17) is uniformly convergent, that is

$$\lim_{m\to\infty} \sum_{i=1}^{m} \Omega_\circ^{i-1} Z_\circ = \sum_{i=1}^{\infty} \Omega_\circ^{i-1} Z_\circ = (E - \Omega_\circ)^{-1} Z_\circ \qquad \text{...........(1.18)}$$

and then the relation (1.18) as certain on the convergence of sequences of the
functions $[x_m(t,x_\circ,y_\circ), y_m(t,x_\circ,y_\circ)]$on the domain (1.6).

Let

$$\left.\begin{aligned} \lim_{m\to\infty} x_m(t,x_\circ,y_\circ) &= x^\circ(t,x_\circ,y_\circ) \\ \lim_{m\to\infty} y_m(t,x_\circ,y_\circ) &= y^\circ(t,x_\circ,y_\circ) \end{aligned}\right\} \qquad \text{........(1.19)}$$

since each of the sequences of the functions (1.4),(1.5) are continuous and periodic in t of period T then the limiting of (1.4) and (1.5) are continuous and periodic in t of period T and

$$x^\circ(t,x_\circ,y_\circ) = x(t,x_\circ,y_\circ) \;, \; y^\circ(t,x_\circ,y_\circ) = y(t,x_\circ,y_\circ)$$

Thus by lemma 1 and the relation (1.19), the inequality (1.9) is satisfied for m = 0,1,2,……

Now we prove that $x(t,x_\circ,y_\circ)$, $y(t,x_\circ,y_\circ)$ are unique solution for the system (1) by contradiction .

We assume that there is two different solutions $x(t,x_\circ,y_\circ)$, $y(t,x_\circ,y_\circ)$ and $\hat{x}(t,x_\circ,y_\circ)$, $\hat{y}(t,x_\circ,y_\circ)$ for the system (1) which are defined, continuous and periodic in t of period T .

That is:

$$\hat{x}(t,x_\circ,y_\circ)=x_\circ e^{At}+\int_0^t e^{A(t-s)}[B(s)\,\hat{x}(s,x_\circ,y_\circ)+f(s,\hat{x}(s,x_\circ,y_\circ),\hat{y}(s,x_\circ,y_\circ))-$$

$$-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)\,\hat{x}(s,x_\circ,y_\circ)+f(s,\hat{x}(s,x_\circ,y_\circ),\hat{y}(s,x_\circ,y_\circ))]ds]ds \quad ....(1.20)$$

$$\hat{y}(t,x_\circ,y_\circ)=y_\circ e^{Ct}+\int_0^t e^{C(t-s)}[D(s)\,\hat{y}(s,x_\circ,y_\circ)+g(s,\hat{x}(s,x_\circ,y_\circ),\hat{y}(s,x_\circ,y_\circ))-$$

$$-\frac{C}{e^{CT}-E}\int_0^T e^{C(T-s)}[D(s)\,\hat{y}(s,x_\circ,y_\circ)+g(s,\hat{x}(s,x_\circ,y_\circ),\hat{y}(s,x_\circ,y_\circ))]ds]ds \quad ....(1.21)$$

Now from the integral equations (1.7), (1.8) and (1.20), (1.21) we have

$$\|x(t,x_\circ,y_\circ)-\hat{x}(t,x_\circ,y_\circ)\|\le[\frac{T}{2}QH+QC_2]\|x(t,x_\circ,y_\circ)-\hat{x}(t,x_\circ,y_\circ)\|+$$

$$+QC_3\|y(t,x_\circ,y_\circ)-\hat{y}(t,x_\circ,y_\circ\| \quad ....(1.22)$$

also

$$\|y(t,x_\circ,y_\circ)-\hat{y}(t,x_\circ,y_\circ)\|\le R\delta_2\|x(t,x_\circ,y_\circ)-\hat{x}(t,x_\circ,y_\circ)\|+$$

$$+[\frac{T}{2}RJ+R\delta_3]\|y(t,x_\circ,y_\circ)-\hat{y}(t,x_\circ,y_\circ)\| \quad ....(1.23)$$

From (1.22),(1.23) we find that

$$\begin{pmatrix}\|x(t,x_\circ,y_\circ)-\hat{x}(t,x_\circ,y_\circ)\|\\ \|y(t,x_\circ,y_\circ)-\hat{y}(t,x_\circ,y_\circ)\|\end{pmatrix}\le\Omega_\circ\begin{pmatrix}\|x(t,x_\circ,y_\circ)-\hat{x}(t,x_\circ,y_\circ)\|\\ \|y(t,x_\circ,y_\circ)-\hat{y}(t,x_\circ,y_\circ)\|\end{pmatrix} \quad ....(1.24)$$

by continuation this process we have

$$\begin{pmatrix}\|x(t,x_\circ,y_\circ)-\hat{x}(t,x_\circ,y_\circ)\|\\ \|y(t,x_\circ,y_\circ)-\hat{y}(t,x_\circ,y_\circ)\|\end{pmatrix}\le\Omega_\circ^m\begin{pmatrix}\|x(t,x_\circ,y_\circ)-\hat{x}(t,x_\circ,y_\circ)\|\\ \|y(t,x_\circ,y_\circ)-\hat{y}(t,x_\circ,y_\circ)\|\end{pmatrix}$$

but from the condition (12) we get $\Omega_\circ^m \to 0$ when $m\to\infty$ and thus we have

$x(t,x_\circ,y_\circ)=\hat{x}(t,x_\circ,y_\circ)$, $y(t,x_\circ,y_\circ)=\hat{y}(t,x_\circ,y_\circ)$ that is each of $x(t,x_\circ,y_\circ)$, $y(t,x_\circ,y_\circ)$ are unique solution for the system (1)

Section Two : The existence of the periodic solution for the system (1)

The problem of the existence of the periodic solution for the system (1) is uniquely connected with the existence of zeros the functions $\Delta_1^*=\Delta_1^*(x_\circ,y_\circ)$, $\Delta_2^*=\Delta_2^*(x_\circ,y_\circ)$ where:

$$\Delta_1(x_\circ,y_\circ)=\left(\frac{A}{e^{AT}-E}\right)\int_0^T e^{A(T-t)}[B(t)x^\circ(t,x_\circ,y_\circ)+$$

$$+f(t,x^\circ(t,x_\circ,y_\circ),y^\circ(t,x_\circ,y_\circ))]dt \qquad ....(2.1)$$

and

$$\Delta_2(x_\circ,y_\circ)=\left(\frac{C}{e^{CT}-E}\right)\int_0^T e^{C(T-t)}[D(t)y^\circ(t,x_\circ,y_\circ)+$$

$$+g(t,x^\circ(t,x_\circ,y_\circ),y^\circ(t,x_\circ,y_\circ))]dt \qquad ....(2.2)$$

Since these functions is approximately determined from the sequence of functions

$$\Delta_{1m}(x_\circ,y_\circ)=\left(\frac{A}{e^{AT}-E}\right)\int_0^T e^{A(T-t)}[B(t)x_m(t,x_\circ,y_\circ)+$$

$$+f(t,x_m(t,x_\circ,y_\circ),y_m(t,x_\circ,y_\circ))]dt \qquad ....(2.3)$$

and

$$\Delta_{2m}(x_\circ,y_\circ)=\left(\frac{C}{e^{CT}-E}\right)\int_0^T e^{C(T-t)}[D(t)y_m(t,x_\circ,y_\circ)+$$

$$+g(t,x_m(t,x_\circ,y_\circ),y_m(t,x_\circ,y_\circ))]dt \qquad ....(2.4)$$

where m=0,1,2,…….

Theorem 2:-

Under the assumptions of theorem 1 we have the following inequalities:

$$\|\Delta_1(x_\circ,y_\circ)-\Delta_{1m}(x_\circ,y_\circ)\|\le\left\langle\left(N_1^*Q(TH+C_2)\quad N_1^*QC_3\right),\Omega_\circ^m(E-\Omega_\circ)^{-1}Z_\circ\right\rangle$$

$$=\rho_m^* \qquad ....(2.5)$$

$$\|\Delta_2(x_\circ,y_\circ)-\Delta_{2m}(x_\circ,y_\circ)\|\le\left\langle\left(N_2^*R\delta_2\quad N_2^*R(TJ+\delta_3)\right),\Omega_\circ^m(E-\Omega_\circ)^{-1}Z_\circ\right\rangle$$

$$=\gamma_m^* \qquad ....(2.6)$$

Where $N_1^* = \dfrac{\|A\|}{e^{\|A\|T} - \|E\|}$, $N_2^* = \dfrac{\|C\|}{e^{\|C\|T} - \|E\|}$, ρ_m^*, γ_m^* are positive numbers , with $m \geq 0, x_\circ \in D_f$, $y_\circ \in D_g$
and $\langle . \rangle$ denote to the scalar product in the Euclidean space R^n .

Proof :-

By equations (2.1), (2.3) and the condition (1.9) we have

$$\|\Delta_1(x_\circ, y_\circ) - \Delta_{1m}(x_\circ, y_\circ)\| \leq$$

$$\leq \left(\frac{\|A\|}{e^{\|A\|T} - \|E\|} \right) [Q(TH + C_2)\|x^\circ(t, x_\circ, y_\circ) - x_m(t, x_\circ, y_\circ)\| +$$

$$+ Q C_3 \|y^\circ(t, x_\circ, y_\circ) - y_m(t, x_\circ, y_\circ)\|]$$

$$\leq \left\langle \left(N_1^* Q(TH + C_2) \quad N_1^* Q C_3 \right), \Omega_\circ^m (E - \Omega_\circ)^{-1} Z_\circ \right\rangle = \rho_m^*$$

also by using equations (2.2), (2.4) we have

$$\|\Delta_2(x_\circ, y_\circ) - \Delta_{2m}(x_\circ, y_\circ)\| \leq$$

$$\leq \left(\frac{\|C\|T}{e^{\|C\|T} - \|E\|} \right) [R(TJ + \delta_3)\|y^\circ(t, x_\circ, y_\circ) - y_m(t, x_\circ, y_\circ)\| +$$

$$+ R\delta_2 \|x^\circ(t, x_\circ, y_\circ) - x_m(t, x_\circ, y_\circ)\|]$$

$$\leq \left\langle \left(N_2^* R\delta_2 \quad N_2^* R(TJ + \delta_3) \right), \Omega_\circ^m (E - \Omega_\circ)^{-1} Z_\circ \right\rangle = \gamma_m^*$$

Remark 1 :-

When $R^n = R^1$, i.e when x and y are a scalar points, thus we have

Theorem 3:-

Let the functions $f(t, x, y)$ and $g(t, x, y)$ of the system (1) are defined on the intervals [a, b] ,[c, d] in R^1 and for any integer $m \geq 1$ the functions (2.3), (2.4) satisfies the inequalities:

$$\left.\begin{array}{l}\min \Delta_{1m}(x_\circ, y_\circ) \le -\rho_m^* \\ a+h \le x_\circ \le b-h \\ c+h^* \le y_\circ \le d-h^* \\ \max \Delta_{1m}(x_\circ, y_\circ) \ge \rho_m^* \\ a+h \le x_\circ \le b-h \\ c+h^* \le y_\circ \le d-h^*\end{array}\right\} \quad \text{........(2.7)}$$

$$\left.\begin{array}{l}\min \Delta_{2m}(x_\circ, y_\circ) \le -\gamma_m^* \\ a+h \le x_\circ \le b-h \\ c+h^* \le y_\circ \le d-h^* \\ \max \Delta_{2m}(x_\circ, y_\circ) \ge \gamma_m^* \\ a+h \le x_\circ \le b-h \\ c+h^* \le y_\circ \le d-h^*\end{array}\right\} \quad \text{........(2.8)}$$

For all

$$N_1^* = \frac{\|A\|}{e^{\|A\|T} - \|E\|}, \ \rho_m^* = \left\langle \left(N_1^* Q(TH + C_2) \quad N_1^* QC_3 \right), \Omega_\circ^m (E - \Omega_\circ)^{-1} Z_\circ \right\rangle, m \ge 0$$

$$N_2^* = \frac{\|C\|}{e^{\|C\|T} - \|E\|}, \ \gamma_m^* = \left\langle \left(N_2^* R\delta_2 \quad N_2^* R(TJ + \delta_3) \right), \Omega_\circ^m (E - \Omega_\circ)^{-1} Z_\circ \right\rangle, m \ge 0$$

$$h^* = \frac{T}{2} N^* + R\delta_1 \ , \ h = \frac{T}{2} N + QC_1 \ , \quad N^* = R^2 J \sigma_\circ \ , \ N = Q^2 H \delta_\circ$$

then the system (1) have periodic solutions $x = x(t, x_\circ, y_\circ), y = y(t, x_\circ, y_\circ)$

for $x_\circ \in [a+h, b-h]$ and $y_\circ \in [c+h^*, d-h^*]$**.**

Proof :-

Let x_1, x_2 be any two points in the interval $[a+h, b-h]$ *and* y_2, y_1 be any two points in the interval $[c+h^*, d-h^*]$ such that

$$\left.\begin{aligned}
&\Delta_{1m}(x_1,y_1)=\min\Delta_{1m}(x_\circ,y_\circ)\\
&a+h\le x_\circ\le b-h\\
&c+h^*\le y_\circ\le d-h^*\\
&\Delta_{1m}(x_2,y_2)=\max\Delta_{1m}(x_\circ,y_\circ)\\
&a+h\le x_\circ\le b-h\\
&c+h^*\le y_\circ\le d-h^*
\end{aligned}\right\}\qquad ....(2.9)$$

$$\left.\begin{aligned}
&\Delta_{2m}(x_1,y_1)=\min\Delta_{2m}(x_\circ,y_\circ)\\
&a+h\le x_\circ\le b-h\\
&c+h^*\le y_\circ\le d-h^*\\
&\Delta_{2m}(x_2,y_2)=\max\Delta_{2m}(x_\circ,y_\circ)\\
&a+h\le x_\circ\le b-h\\
&c+h^*\le y_\circ\le d-h^*
\end{aligned}\right\}\qquad ....(2.10)$$

by using the inequalities (2.5),(2.6),(2.7),(2.8) we have

$$\left.\begin{aligned}
&\Delta_1(x_1,y_1)=\Delta_{1m}(x_1,y_1)+(\Delta_1(x_1,y_1)-\Delta_{1m}(x_1,y_1))<0\\
&\Delta_1(x_2,y_2)=\Delta_{1m}(x_2,y_2)+(\Delta_1(x_2,y_2)-\Delta_{1m}(x_2,y_2))>0
\end{aligned}\right\}\qquad ......(2.11)$$

$$\left.\begin{aligned}
&\Delta_2(x_1,y_1)=\Delta_{2m}(x_1,y_1)+(\Delta_2(x_1,y_1)-\Delta_{2m}(x_1,y_1))<0\\
&\Delta_2(x_2,y_2)=\Delta_{2m}(x_2,y_2)+(\Delta_2(x_2,y_2)-\Delta_{2m}(x_2,y_2))>0
\end{aligned}\right\}\qquad ......(2.12)$$

and from the continuity of the functions $\Delta_1(x_\circ,y_\circ)$, $\Delta_2(x_\circ,y_\circ)$ and the inequalities (2.11), (2.12) then there exist an isolated singular point $(x^\circ,y^\circ)=(x_\circ,y_\circ)$, $x^\circ\in[x_1,x_2]$ *and* $y^\circ\in[y_1,y_2]$ where $\Delta_1(x_\circ,y_\circ)=0,\Delta_2(x_\circ,y_\circ)=0$.

That is $x=(t,x_\circ,y_\circ)$ and $y=(t,x_\circ,y_\circ)$ a periodic solutions of the system (1), for $x_\circ\in[a+h,b-h]$, $y_\circ\in[c+h^*,d-h^*]$.

Theorem 4 :-

Let

$\Delta_1 : D_f \times D_g \to R^n$,

$$\Delta_1(x_\circ, y_\circ) = \left(\frac{A}{e^{AT} - E}\right)\int_0^T e^{A(T-t)}[B(t)x^\circ(t, x_\circ, y_\circ) +$$

$$+ f(t, x^\circ(t, x_\circ, y_\circ), y^\circ(t, x_\circ, y_\circ))]dt \quad ....(2.13)$$

and

$\Delta_2 : D_f \times D_g \to R^n$,

$$\Delta_2(x_\circ, y_\circ) = \left(\frac{C}{e^{CT} - E}\right)\int_0^T e^{C(T-t)}[D(t)y^\circ(t, x_\circ, y_\circ) +$$

$$+ g(t, x^\circ(t, x_\circ, y_\circ), y^\circ(t, x_\circ, y_\circ))]dt \quad ....(2.14)$$

where $x^\circ(t, x_\circ, y_\circ)$ and $y^\circ(t, x_\circ, y_\circ)$ are the limiting functions of the periodic sequence (1.4) and (1.5), then the following inequalities

$$\|\Delta_1(x_\circ, y_\circ)\| \le M_7 \quad ....(2.15)$$

where

$$M_7 = N_1 QH(\delta_\circ QM_3 + QC_1 M_3) + N_1^* QC_1 \ , \ M_3 = (1 - \frac{T}{2} QH)^{-1}$$

$$\|\Delta_2(x_\circ, y_\circ)\| \le M_8 \quad ....(2.16)$$

and $M_8 = N_2 RJ(\sigma_\circ RM_5 + R\delta_1 M_5) + N_2^* R\delta_1 \ , M_5 = (1 - \frac{T}{2} RJ)^{-1}$

$$\|\Delta_1(x_\circ^1, y_\circ^1) - \Delta_1(x_\circ^2, y_\circ^2)\| \le [N_1^* E_5 W_1 W_2 (1 - E_4) + N_1^* E_2 E_3 W_1 W_2]\|x_\circ^1 - x_\circ^2\| Q +$$

$$+ [N_1^* E_2 E_5 W_1 W_2 + N_1^* E_2 (1 + E_2 E_3 W_1 W_2)(1 - E_4)^{-1}]\|y_\circ^1 - y_\circ^2\| R \quad ....(2.17)$$

$$\|\Delta_2(x_\circ^1, y_\circ^1) - \Delta_2(x_\circ^2, y_\circ^2)\| \le [N_2^* E_3 W_1 W_2 (1 - E_4) + N_2^* E_3 E_4 W_1 W_2]\|x_\circ^1 - x_\circ^2\| Q +$$

$$+ [N_2^* E_2 E_3 W_1 W_2 + N_2^* E_4 (1 + E_2 E_3 W_1 W_2)(1 - E_4)^{-1}]\|y_\circ^1 - y_\circ^2\| R \quad ....(2.18)$$

satisfies for $x_\circ, x_\circ^1, x_\circ^2 \in D_f, y_\circ, y_\circ^1, y_\circ^2 \in D_g$

Where

$$N_1 = \frac{\|A\| T}{e^{\|A\| T} - \|E\|}, N_2 = \frac{\|C\| T}{e^{\|C\| T} - \|E\|}, N_1^* = \frac{\|A\|}{e^{\|A\| T} - \|E\|}, N_2^* = \frac{\|C\|}{e^{\|C\| T} - \|E\|}$$

$$E_1 = [\frac{T}{2} QH + QC_2], \ E_2 = QC_3, E_3 = R\delta_2 \ , E_4 = [\frac{T}{2} RJ + R\delta_3], E_5 = Q(TH + C_2)$$

$$W_1 = [(1 - E_1)(1 - E_4)]^{-1}, W_2 = (1 - E_2 E_3 W_1)^{-1}$$

Proof :-

From the properties of the functions $x^{\circ}(t,x_{\circ},y_{\circ})$, $y^{\circ}(t,x_{\circ},y_{\circ})$ that theorem 1 then each of the functions $\Delta_1 = \Delta_1(x_{\circ},y_{\circ})$, $\Delta_2 = \Delta_2(x_{\circ},y_{\circ})$, $x_{\circ} \in D_f$, $y_{\circ} \in D_g$ continuous and bounded by non negative constants M_7, M_8, and from the relation (2.13) we find that:

$$\|\Delta_1(x_{\circ},y_{\circ})\| \le \frac{\|A\|T}{e^{\|A\|T} - \|E\|} QH \|x^{\circ}(t,x_{\circ},y_{\circ})\| + \frac{\|A\|}{e^{\|A\|T} - \|E\|} QC_1 \quad ....(2.19)$$

since that $x^{\circ}(t,x_{\circ},y_{\circ})$ satisfy the integral equation (1.7) and by using lemma1 we have:

$$\|x^{\circ}(t,x_{\circ},y_{\circ})\| \le \delta_{\circ} QM_3 + QC_1 M_3 \quad ....(2.20)$$

and

$$\|\Delta_2(x_{\circ},y_{\circ})\| \le \frac{\|C\|T}{e^{\|C\|T} - \|E\|} RJ \|y^{\circ}(t,x_{\circ},y_{\circ})\| + \frac{\|C\|}{e^{\|C\|T} - \|E\|} R\delta_1 \ , \quad ....(2.21)$$

since that $y^{\circ}(t,x_{\circ},y_{\circ})$ satisfy the integral equation (1.8) and by using lemma1 we find that

$$\|y^{\circ}(t,x_{\circ},y_{\circ})\| \le \sigma_{\circ} RM_5 + R\delta_1 M_5 \quad .....(2.22)$$

and from the relation (2.13) we get:

$$\|\Delta_1(x_{\circ}^1,y_{\circ}^1) - \Delta_1(x_{\circ}^2,y_{\circ}^2)\| \le N_1^* E_5 \|x^{\circ}(t,x_{\circ}^1,y_{\circ}^1) - x^{\circ}(t,x_{\circ}^2,y_{\circ}^2)\| +$$

$$+ N_1^* E_2 \|y^{\circ}(t,x_{\circ}^1,y_{\circ}^1) - y^{\circ}(t,x_{\circ}^2,y_{\circ}^2)\| \quad ....(2.23)$$

where

$$x(t,x_{\circ}^k,y_{\circ}^k) = x_{\circ}^k e^{At} + \int_0^t e^{A(t-s)}[B(s)x(s,x_{\circ}^k,y_{\circ}^k) + f(s,x(s,x_{\circ}^k,y_{\circ}^k),y(s,x_{\circ}^k,y_{\circ}^k)) -$$

$$-\frac{A}{e^{AT} - E}\int_0^T e^{A(T-s)}[B(s)\,x(s,x_{\circ}^k,y_{\circ}^k) + f(s,x(s,x_{\circ}^k,y_{\circ}^k),y(s,x_{\circ}^k,y_{\circ}^k))]ds]ds \quad ....(2.24)$$

$$y(t,x_{\circ}^k,y_{\circ}^k) = y_{\circ}^k e^{Ct} + \int_0^t e^{C(t-s)}[D(s)y(s,x_{\circ}^k,y_{\circ}^k) + g(s,x(s,x_{\circ}^k,y_{\circ}^k),y(s,x_{\circ}^k,y_{\circ}^k)) -$$

$$-\frac{C}{e^{CT} - E}\int_0^T e^{C(T-s)}[D(s)\,y(s,x_{\circ}^k,y_{\circ}^k) + g(s,x(s,x_{\circ}^k,y_{\circ}^k),y(s,x_{\circ}^k,y_{\circ}^k))]ds]ds \quad ....(2.25)$$

where k =1,2.

since $x^{\circ}(t,x_{\circ},y_{\circ}), y^{\circ}(t,x_{\circ},y_{\circ})$ satisfies the two equations (1.7),(1.8) on arrangement.

from the relation (2.24) we find that

$$\|x^{\circ}(t,x_{\circ}^{1},y_{\circ}^{1})-x^{\circ}(t,x_{\circ}^{2},y_{\circ}^{2})\| \le (1-E_{1})^{-1}\|x_{\circ}^{1}-x_{\circ}^{2}\|Q+$$

$$+E_{2}(1-E_{1})^{-1}\|y^{\circ}(t,x_{\circ}^{1},y_{\circ}^{1})-y^{\circ}(t,x_{\circ}^{2},y_{\circ}^{2})\| \quad ......(2.26)$$

also from the relation (2.25) we find that

$$\|y^{\circ}(t,x_{\circ}^{1},y_{\circ}^{1})-y^{\circ}(t,x_{\circ}^{2},y_{\circ}^{2})\| \le (1-E_{1})^{-1}\|y_{\circ}^{1}-y_{\circ}^{2}\|R+$$

$$+E_{3}(1-E_{4})^{-1}\|x^{\circ}(t,x_{\circ}^{1},y_{\circ}^{1})-x^{\circ}(t,x_{\circ}^{2},y_{\circ}^{2})\|$$

$$\ldots\ldots\ldots(2.27)$$

by substitutions the inequality (2.27) in the inequality (2.26) we obtain

$$\|x^{\circ}(t,x_{\circ}^{1},y_{\circ}^{1})-x^{\circ}(t,x_{\circ}^{2},y_{\circ}^{2})\| \le (1-E_{1})^{-1}\|x_{\circ}^{1}-x_{\circ}^{2}\|Q+E_{2}W_{1}\|y_{\circ}^{1}-y_{\circ}^{2}\|R+$$

$$+E_{2}E_{3}W_{1}\|x^{\circ}(t,x_{\circ}^{1},y_{\circ}^{1})-x^{\circ}(t,x_{\circ}^{2},y_{\circ}^{2})\|$$

since $W_{1}(1-E_{4})=(1-E_{1})^{-1}$ then

$$\|x^{\circ}(t,x_{\circ}^{1},y_{\circ}^{1})-x^{\circ}(t,x_{\circ}^{2},y_{\circ}^{2})\| \le W_{1}W_{2}(1-E_{4})\|x_{\circ}^{1}-x_{\circ}^{2}\|Q+E_{2}W_{1}W_{2}\|y_{\circ}^{1}-y_{\circ}^{2}\|R$$

$$.........(2.28)$$

by substitutions the inequality (2.28) in the inequality (2.27) we obtain

$$\|y^{\circ}(t,x_{\circ}^{1},y_{\circ}^{1})-y^{\circ}(t,x_{\circ}^{2},y_{\circ}^{2})\| \le [1+E_{2}E_{3}W_{1}W_{2}](1-E_{4})^{-1}\ \|y_{\circ}^{1}-y_{\circ}^{2}\|R+$$

$$+E_{3}W_{1}W_{2}\|x_{\circ}^{1}-x_{\circ}^{2}\ \|Q \quad ....(2.29)$$

by substitutions the two inequalities (2.28),(2.29) in the inequality (2.23) we obtain (2.17).

also by the relation (2.14) we find that

$$\|\Delta_{2}(x_{\circ}^{1},y_{\circ}^{1})-\Delta_{2}(x_{\circ}^{2},y_{\circ}^{2})\| \le N_{2}^{*}E_{3}\|x^{\circ}(t,x_{\circ}^{1},y_{\circ}^{1})-x^{\circ}(t,x_{\circ}^{2},y_{\circ}^{2})\|+$$

$$+N_{2}^{*}E_{4}\|y^{\circ}(t,x_{\circ}^{1},y_{\circ}^{1})-y^{\circ}(t,x_{\circ}^{2},y_{\circ}^{2})\| \quad ....(2.30)$$

by substitutions the inequalities (2.28),(2.29) in the inequality (2.30) we obtain (2.18)

Remark 2 :-

By [4], we conclude that theorem 4 insures the stability of the solution for the system (1) of non linear differential equations since a slight change in the point $(x_{\circ},y_{\circ})$ leads to a slight change in the functions

$\Delta_{1}=\Delta_{1}(x_{\circ},y_{\circ})$, $\Delta_{2}=\Delta_{2}(x_{\circ},y_{\circ})$.

REFERENCES

1- Butris, R. N. and Saied, EL. M. *On periodic solutions for certain systems of integro - differential equations*, Iraq, Mosul, J. of Educ. And Sci, Vol.31, p.p. 80-90, (1998).

2- Butris, R. N. and AL –Ameen , M .S. *The existence of periodic solutions for nonlinear systems of integro - differential equations,* Iraq, Mosul, J. of Educ. And Sci Vol. 34, 37-48, (1999).

3- Martynuk, D. I. *Periodic solutions of second – order nonlinear differential equation* Ukrain, Math. J. No.4, pp. 125-132, (1967).

4- Mitropolsky, Yu. A. and Martynuk, D. I. *Periodic solutions for the oscillations systems with retarded arguments*, Ukrain, Kiev, (1979).

5- Naima, D. M. *Periodic solutions of non autonomous second - order differential equations boundary value problems – numerical solutions differential equations,* nonlinear studies, Vol. 6, Issue.8-14 , (1999).

6- Perestyuk, N. A. *The periodic solution for nonlinear systems of differential equation,* Ukrain, Kiev, univ of kiev, Math. and Meca. J. No. 5, (1971).

7- Samoilenko, A . M. *A numerical-analytic methods for investigations of periodic systems of ordinary differential equations* I , II , Ukrain, Kiev, Math. J. No.4,5 , p.p. (82-93) ,(50-59) ,(1965,1966).