Reformulation of the Generalized Curvature by Using Decomposition Technique

Tahir H. Ismail
Dept. of Mathematics
College of Computer Science & Math
University of Mosul

Ibrahim O. Hamad
Dept. Of Mathematics
College of Science
Salahaddin University-Hawler

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الملخص

في هذا البحث. نحاول إنشاء صيغة جديدة لتقوس معمم للمنحني المستوي $\gamma(t)$ معتمدين على نظرية التحال وذلك للحصول على تقوس معمم للمنحنيات جبرية مستخدمين بعض مفاهيم التحليل غير القياسي الذي أوجده Robinson, A. ووضعه Nelson, E. بأسلوب منطقي.

ABSTRACT

The aim of this paper is to establish a new formula of generalized curvature discussed with the help of decomposition theorem which is used to obtain generalized curvature for algebraic curves by using some concepts of nonstandard analysis given by Robinson A. and axiomatized by Nelson E.

1-Introduction: -

Among the difference expressions of the generalized curvature, there is one which can be expressed by using only one point infinitely close to a singular point of a standard curve and this will be our consideration in this paper. This is derived from generalized curvature form [10], with the help of decomposition theorem.

The following definitions of nonstandard analysis will be needed throughout this paper. [4], [5], [9], [11], [12], [15], [16], [17]

A real number x is called unlimited if and only if $|x| > r$ for all positive standard real numbers, otherwise it is called limited.

The set of all unlimited real numbers denoted by $\overline{\mathbf{R}}$, and the set of all limited real numbers denoted by $\underline{\mathbf{R}}$

A real number x is called infinitesimal if and only if $|x| < r$ for all positive standard real numbers r .

Two real numbers x and y are said to be infinitely close if and only if $x - y$ is infinitesimal and denoted by $x \cong y$.

If x is a limited number in $\mathbf{R}$, then it is infinitely close to a unique standard real number, this unique number is called the standard part of x or shadow of x denoted by $st(x)$ or ${}^{0}x$.

Every set or element defined in a classical mathematics is called standard.

Any set or formula which does not involve new predicates "standard, infinitesimals, limited,…etc" is called internal, otherwise is called external.

Let $\alpha : \mathbf{I} \to \mathbf{R}^3$, $\mathbf{I} = (a,b)$, be a curve parametrized by an arc length s, and its tangent vector be $\alpha\,'$ which has a unit length, then the measure of the ratio of change of the angle with neighboring tangents made with the tangent at s is known as a curvature of the curve α at s ,and it is given by $|g''(s)|$ and denoted by $\kappa(s)$

Theorem 1.1 [10]

Let $A = \gamma(t)$ be a standard point on the planar curve γ, and let B and C be two points infinitely close to the point A, then

$$\frac{\tan \hat{A}}{\left\|\overrightarrow{BC}\right\|} = \frac{\overrightarrow{AC} \times \overrightarrow{AB}}{\left\|\overrightarrow{BC}\right\|^3} \cong \frac{\overrightarrow{BC} \times \overrightarrow{AB}}{\left\|\overrightarrow{BC}\right\|^3}.$$

Theorem 1.2 [10]

Let A be a standard point on the curve γ, and let B and C be two points infinitely close to the point A, then the generalized curvature κ_G of the curve γ at the point A is given by: $\kappa_G \cong \frac{\left|\tan \hat{A}\right|}{\left\|\overrightarrow{BC}\right\|} \cong \frac{\left|\overrightarrow{AC} \times \overrightarrow{AB}\right|}{\left\|\overrightarrow{BC}\right\|^3}$.

2- Decomposition Technique for Calculating the Generalized Curvature

According to the applications of decomposition theorem to algebraic equations, we give an expression of the generalized curvature, which does not involve the parametric but a point which infinitely close to the considered standard part.

Theorem 2.1(Decomposition Technique Theorem)

If M is a limited point of $\mathbf{R}^2$ whose shadow is ${}^{o}M = M_o$, then for $M \neq M_o$ there exist two standard vectors $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ linearly independent and two real infinitesimal numbers ε_1 and ε_2 such that $\overrightarrow{M} \cong \overrightarrow{M}_o + \varepsilon_1 \overrightarrow{V_1} + \varepsilon_1 \varepsilon_2 \overrightarrow{V_2}$.

Proof:

Since ${}^o M = M_o$ then there exist an infinitesimal $\vec{\eta}$ such that

$$\overrightarrow{M} = \overrightarrow{M}_o + \vec{\eta} \qquad \cdots \textbf{(2.1.1)}$$

where $M, M_o, \eta \in \mathbf{R}^2$.

Put $\varepsilon_1 = \left\| \overrightarrow{M} - \overrightarrow{M}_o \right\|$ and $\overrightarrow{P_1} = \dfrac{\overrightarrow{M} - \overrightarrow{M}_o}{\varepsilon_1}$ $\qquad \cdots \textbf{(2.1.2)}$

then $\overrightarrow{P_1}$ is a unit direction vector.

Also put $\overrightarrow{V_1} = {}^o\overrightarrow{P_1}$ then there exist an infinitesimal $\vec{\zeta}$ such that

$$\overrightarrow{P_1} = \overrightarrow{V}_1 + \vec{\zeta} \qquad \cdots \textbf{(2.1.3)}$$

Put $\varepsilon_2 = \left\| \overrightarrow{P_1} - \overrightarrow{V}_1 \right\|$, $\overrightarrow{P_2} = \dfrac{\overrightarrow{P_1} - \overrightarrow{V}_1}{\varepsilon_2}$, and $\overrightarrow{V_2} = {}^o\overrightarrow{P_2}$,

then there exist an infinitesimal ξ such that $\overrightarrow{P_2} = \overrightarrow{V}_2 + \vec{\xi}$ $\quad \cdots \textbf{(2.1.4)}$

Combine equations $\textbf{(2.1.2)} - \textbf{(2.1.4)}$ and substitute them in equation $\textbf{(2.1.1)}$ we get that $\overrightarrow{M} = \overrightarrow{M}_o + \varepsilon_1 \overrightarrow{V_1} + \varepsilon_1 \varepsilon_2 \overrightarrow{V_2} + \varepsilon_1 \varepsilon_2 \vec{\xi}$

Thus $\quad \overrightarrow{M} \cong \overrightarrow{M}_o + \varepsilon_1 \overrightarrow{V_1} + \varepsilon_1 \varepsilon_2 \overrightarrow{V_2}$.

Corollary 2.2

If M is limited point of $\mathbf{R}^2$ whose shadow is ${}^o M = M_o$, then for ${}^o M = M_o$ there exist two real infinitesimal numbers ε_1 and ε_2 such that $\overrightarrow{M} \cong \overrightarrow{M}_o + \varepsilon_1 \overrightarrow{V_1} + \varepsilon_1 \varepsilon_2 \overrightarrow{V_2}$ where $\left\{ \overrightarrow{V_1}, \overrightarrow{V_2} \right\}$ is a standard basis of $\mathbf{R}^2$.

Proof:

Since $\overrightarrow{{}^o M} = \overrightarrow{M}_o$ then there exist an infinitesimal $\vec{\eta}$ such that $\overrightarrow{M} = \overrightarrow{M}_o + \vec{\eta}$

and $\overrightarrow{M}, \overrightarrow{M}_o, \vec{\eta} \in \mathbf{R}^2$.

Since $\left\{ \overrightarrow{V_1}, \overrightarrow{V_2} \right\}$ is a standard basis of $\mathbf{R}^2$, then there exist two standard real numbers *a* and *b*, and two real infinitesimals α and β not both zero, such that

$$\overrightarrow{M} = a\overrightarrow{V_1} + b\overrightarrow{V_2} + \alpha\overrightarrow{V_1} + \beta\overrightarrow{V_2}.$$

Now

if $\alpha \neq 0$, then $\overrightarrow{M} = \overrightarrow{M}_o + \varepsilon_1 \overrightarrow{V_1} + \varepsilon_1\varepsilon_2 \overrightarrow{V_2}$, where $\varepsilon_1 = \alpha$ and $\varepsilon_2 = \dfrac{\beta}{\alpha}$

if $\beta \neq 0$, then $\overrightarrow{M} = \overrightarrow{M}_o + \varepsilon_1 \overrightarrow{\varepsilon_2 V_1} + \varepsilon_2 \overrightarrow{V_2}$, where $\varepsilon_1 = \dfrac{\alpha}{\beta}$ and $\varepsilon_2 = \beta$.

Remark:

1- The choice of *a* and *b* to be standard and α, β to be infinitesimals is necessary, since $\overrightarrow{M}_o$ is standard and $\vec{\eta}$ is infinitesimal.

2- If $\{\overrightarrow{V_1}, \overrightarrow{V_2}\}$ is a standard usual basis of $\mathbf{R}^2$, then

$$\overrightarrow{M} = (x, y) = \overrightarrow{M}_o + \varepsilon_1 \overrightarrow{V_1} + \varepsilon_1\varepsilon_2 \overrightarrow{V_2} = (x_o + \varepsilon_1, y_o + \varepsilon_1\varepsilon_2).$$

3- If $o(x - x_o) = o(y - y_o)$, then $\overrightarrow{M} = (x, y) = (x_o + \varepsilon, y_o + \varepsilon)$, where *o* is denoted to the small oh

4- For any basis $B = \{\overrightarrow{V_1}, \overrightarrow{V_2}\}$, the matrix form of the decomposition theorem in $\mathbf{R}^2$ is given by: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \begin{pmatrix} a+\alpha \\ b+\beta \end{pmatrix}$, that is $\vec{X} = B\vec{C}$ where $\overrightarrow{V}_1 = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}$, $\overrightarrow{V}_2 = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}$ and $\vec{C}$ is the constant matrix $\begin{pmatrix} a+\alpha \\ b+\beta \end{pmatrix}$, and B is the constant matrix $\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$.

Theorem 2.3(General Decomposition Technique Theorem)

If M is limited point of $\mathbf{R}^n$ whose shadow is ${}^oM = M_o$, then for $M \neq M_o$ there exist n standard vectors $\overrightarrow{V_1}, \overrightarrow{V}_2, \cdots, \overrightarrow{V}_n$ which are linearly independent, and n real infinitesimal numbers $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ such that

$$\overrightarrow{M} \cong \overrightarrow{M}_o + \varepsilon_1 \overrightarrow{V_1} + \varepsilon_1\varepsilon_2 \overrightarrow{V_2} + \cdots + \varepsilon_1\varepsilon_2 \cdots \varepsilon_n \overrightarrow{V_n}.$$

Theorem 2.4(Decomposition Version of Generalized Curvature)

Let M_o be a standard point of a standard curve γ in C^∞, and let $\overrightarrow{M} \cong \overrightarrow{M}_o + \varepsilon_1 \overrightarrow{V_1} + \varepsilon_1\varepsilon_2 \overrightarrow{V_2}$, then the general curvature at M_o is independent of the choice of M and it given by $\quad {}^o\kappa_G = \left| \dfrac{\varepsilon_2}{\varepsilon_1^{\frac{q}{p}-1}} \right|$

Proof:

Let p and q be the respective orders of the first non zero vector derivative $\gamma^{(p)}$ and of the first vector derivative not collinear with $\gamma^{(q)}$.

By using the Taylor development up to the order q, and the decomposition theorem, the point M can be written in two different ways:

$$\overrightarrow{M} \cong \overrightarrow{M}_o + \varepsilon_1\overrightarrow{V_1} + \varepsilon_1\varepsilon_2\overrightarrow{V_2},$$

and

$$\overrightarrow{M} \cong \gamma(t_o) + \cdots + \frac{\gamma^{(p)}(t_o)}{p!}(t-t_o)^p + \cdots + \frac{\gamma^{(q)}(t_o)}{q!}(t-t_o)^q + \eta(t-t_o)^q,$$

where $\eta \cong 0$ Putting $t - t_o = \delta,\ \delta \cong 0$ we get

$$\varepsilon_1\overrightarrow{V_1} + \varepsilon_1\varepsilon_2\overrightarrow{V_2} = \frac{\gamma^{(p)}(t_o)}{p!}\delta^p + \cdots + \frac{\gamma^{(q)}(t_o)}{q!}\delta^q + \eta\delta^q \qquad \cdots(2.4.1)$$

Dividing equation **(3.4.1)** by ε_1 we get

$$\overrightarrow{V_1} + \varepsilon_2\overrightarrow{V_2} = \frac{\gamma^{(p)}(t_o)}{p!}\frac{\delta^p}{\varepsilon_1} + \cdots + \frac{\gamma^{(q)}(t_o)}{q!}\frac{\delta^q}{\varepsilon_1} + \eta\frac{\delta^q}{\varepsilon_1} \qquad \cdots(2.4.2)$$

Since $\frac{\delta^p}{\varepsilon_1}$ is limited and $\frac{\delta^q}{\varepsilon_1} \cong 0 \quad \forall q > p$, then from the decomposition theorem, we have $\varepsilon_1 = \|M - M_o\|$

Therefore

$$\frac{\delta^p}{\varepsilon_1} = \frac{\delta^p}{\left\|\frac{\gamma^p}{p!}\delta^p + \cdots + \frac{\gamma^q}{q!}\delta^q + \eta\delta^q\right\|} = \frac{1}{\left\|\frac{\gamma^p}{p!} + \beta\right\|}, \beta \cong 0$$

Thus $\frac{\delta^p}{\varepsilon_1} \cong \frac{p!}{\|\gamma^p\|}$, which is limited. Taking the shadow of the equation **(2.4.2)** we get $V_1 = \frac{\gamma^p}{\|\gamma^p\|}$ which is a unit vector in $\mathbf{R}^2$. The scalar product of equation **(2.4.1)** by V_1 implies

$$\varepsilon_1 = \frac{\|\gamma^{(p)}\|}{p!}\delta^p + \cdots + \frac{\gamma^{(q)} \bullet \gamma^{(p)}}{q!\|\gamma^{(p)}\|}\delta^q + \xi \quad , \xi \cong 0 \quad .$$

Then

$$\varepsilon_1 \cong \frac{\|\gamma^{(p)}\|}{p!}\delta^p \qquad \cdots(2.4.3)$$

Again, the norm of the cross product of equation **(2.4.1)** by V_1 is implies

$$|\varepsilon_1\varepsilon_2| \cong \frac{\|\gamma^{(q)} \times \gamma^{(p)}\|}{q!\|\gamma^{(p)}\|}\delta^q$$

… **(2.4.4)**

Now raising equation **(2.4.4)** to the power q and equation **(2.4.3)** to the power p and then dividing the obtained two equations we get

$$\frac{(p!)^{\frac{q}{p}}\left(\gamma^{(q)} \times \gamma^{(p)}\right)}{(q!)\|\gamma^{(p)}\|^{\frac{q}{p}+1}} = {}^o\kappa_G \cong \frac{\varepsilon_2}{\varepsilon_1^{\frac{q}{p}-1}}$$

Corollary 2.5

The curvature $K(t)$ of a standard curve at a standard biregular point is given by

$$K(t) = 2\,{}^o\left|\frac{\varepsilon_2}{\varepsilon_1}\right|.$$

Corollary 2.6

Let M_o be a standard point of a standard curve γ in C^∞, and let $\vec{M} \cong \vec{M}_o + \varepsilon_1\vec{V}_1 + \varepsilon_1\varepsilon_2\vec{V}_2$, where $B = \{V_1 = (a_1, b_1), V_2 = (a_2, b_2)\}$ is any standard basis of $\mathbf{R}^2$, then we have the following cases:

(1) If $a_1 \neq 0, b_1 = 0, b_2 \neq 0,$ then ${}^o\kappa_G = \left|\frac{\varepsilon_2 b_2}{\varepsilon_1^{\frac{q}{p}-1} a_1^{\frac{q}{p}}}\right|.$

(2) If V_1 and V_2 are basic unit direction vectors, then $a_i = 1$, whenever $b_i = 0$, and conversely.

(3) For cases other than (1) and (2), κ_G is unlimited or undefined

Proof:

(1) Form the following table of all possible values of (a_1, b_1) and (a_2, b_2) for the basis B we can deduce the result

$a_1 \neq 0$	$b_1 \neq 0$	$a_2 \neq 0$	$b_2 \neq 0$	${}^{o}\kappa_G$	
Yes	Yes	Yes	Yes	$\left\| \frac{b_1 / a_1}{\varepsilon_1^{\frac{q}{p}-1} [a_1^2 + b_1^2]^{\frac{1}{2}}} \right\|$	This is either unlimited or undefined
Yes	Yes	Yes	No		
Yes	Yes	No	Yes		
Yes	No	Yes	Yes	$\left\| \frac{\varepsilon_2 b_2}{\varepsilon_1^{\frac{q}{p}-1} a_1^{\frac{q}{p}}} \right\|$	
Yes	No	No	Yes		
No	Yes	Yes	Yes	$\left\| \frac{1}{\varepsilon_1 \varepsilon_2 a_2} \right\|$	
No	Yes	Yes	No		

(2) Obviously

(3) In this case, the general curvature is the same as that given by Theorem 2.4, but here we give a simple proof as follows:

Since V_1 and V_2 are basic unit direction vectors, then we have

$$M = (x, y) = (x_o, y_o) + \varepsilon_1(1,0) + \varepsilon_1\varepsilon_2(0,1) = (x_o + \varepsilon_1, y_o + \varepsilon_1\varepsilon_2)$$

Therefore such as shown in the figure (2.1) we get

$$\|M - M_o\| = \varepsilon_1\left(1 + \varepsilon_2^2\right)^{\frac{1}{2}} \cong \varepsilon_1 \text{ and } \tan(\vec{T}\big|_{M_o}) = \tan(\theta) = \frac{\varepsilon_1\varepsilon_2}{\varepsilon_1}$$

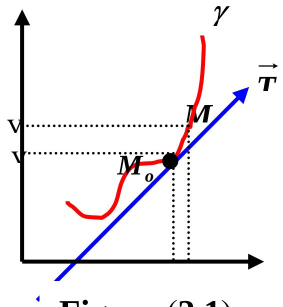

Figure (2.1)

Thus $${}^{o}\kappa_G(\gamma)\big|_{M_o} = \left| \frac{\varepsilon_2}{\varepsilon_1^{\frac{q}{p}-1}} \right|$$

3-Spherical Curvature

Nonstandard studies of curves are now available widely for geometrical and non geometrical applications, [6], [7], [13].

In this section the problem of curvature on spherical Euclidean space is studied. For planar curves [3], [8], we have only two related order derivatives corresponding to a standard curve, while for spherical curves, three related order of derivatives effect the behavior of the curve, as it is shown in Theorem 3.5.

Other studies on spherical curvatures can be found in [1], [2], [14] and [18]. The characterization of this study begins with the fundamental definition of curvature, which is applicable to any curve in all spaces.

Theorem 3.1

Let γ be a standard curve of type C^∞ in $\mathbf{R}^3$ which admits at each point two vector derivatives not coplanar denoted by $\gamma^{(p)}$ and $\gamma^{(q)}$, where p and q are the smallest integers such that $\gamma^{(p)} \times \gamma^{(q)} \neq \mathbf{0}$. Then the generalized curvature of γ is given by

$$ {}^{o}\kappa_G(t) = \frac{(p!)^{\frac{q}{p}} \left|\gamma^{(p)} \times \gamma^{(q)}\right|}{q! \left\|\gamma^{(p)}\right\|^{\frac{q}{p}+1}} \ . $$

Proof:

Let $B = \gamma(t+\alpha)$ and $C = \gamma(t+\alpha\varepsilon)$ where $\alpha, \varepsilon \cong 0$ be two tinfinitely close to the point $A = \gamma(t)$. By Theorem 1.2 , we have

$$ \kappa_G \cong \frac{\left|\tan \hat{A}\right|}{\left\|\overrightarrow{BC}\right\|} \cong \frac{\left|\overrightarrow{AC} \times \overrightarrow{AB}\right|}{\left\|\overrightarrow{BC}\right\|^3} $$

We consider the following cases concerning the point $A = \gamma(t)$

- Case I\ A is a Biregular point:

We have $\left\|\overrightarrow{BC}\right\| \cong |(\beta-\alpha)|\left(x'^2 + y'^2 + z'^2\right)^{\frac{1}{2}}$

Therefore $\left\|\overrightarrow{BC}\right\|^3 \cong \left|(\beta-\alpha)^3\right|\left(x'^2 + y'^2 + z'^2\right)^{\frac{3}{2}}$ $\quad \cdots$**(3.1.1)**

Note that we can obtain two different forms according to the given form of the quantity $(\beta-\alpha)^3$ as follows:

$$ (\beta-\alpha)^3 = \begin{cases} 3\alpha\beta(\alpha-\beta) + (\alpha^3 - \beta^3) \cong 3\alpha\beta(\alpha-\beta) \\ 2\alpha\beta(\beta-\alpha) + (\beta-\alpha)(\alpha^2-\beta^2) \cong 2\alpha\beta(\alpha-\beta) \end{cases} $$

In general, we have $(\beta-\alpha)^3 = c \cdot \alpha\beta(\alpha-\beta)$, where c is an arbitrary standard constant.

For our purpose we take $(\beta-\alpha)^3 = \alpha\beta(\alpha-\beta)$, so

$$ \left\|\overrightarrow{BC}\right\|^3 \cong |\alpha\beta(\alpha-\beta)|\left(x'^2 + y'^2 + z'^2\right)^{\frac{3}{2}} = |\alpha\beta(\alpha-\beta)|\left\|\gamma'(t)\right\| \quad \cdots\textbf{(3.1.2)} $$

And

$$ \overrightarrow{AC} \times \overrightarrow{AB} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \left(\beta x' + \frac{\beta^2}{2} x'' + \delta_2\beta^2\right) & \left(\beta y' + \frac{\beta^2}{2} y'' + \delta_2\beta^2\right) & \left(\beta z' + \frac{\beta^2}{2} z'' + \delta_2\beta^2\right) \\ \left(\alpha x' + \frac{\alpha^2}{2} x'' + \delta_1\alpha^2\right) & \left(\alpha y' + \frac{\alpha^2}{2} y'' + \delta_1\alpha^2\right) & \left(\alpha z' + \frac{\alpha^2}{2} z'' + \delta_1\alpha^2\right) \end{vmatrix} $$

$$\cong \frac{\alpha\beta(\alpha-\beta)}{2}\left((y'z''-z'y'')e_1+(z'x''-x'z'')e_2+(x'y''-y'x'')e_3\right) \quad \cdots(3.1.3)$$

From equation **(3.1.2)** and **(3.1.3)** we get the result

Case II\ A is an Only Regular point:

Since $A=\gamma(t_o)$ is an only regular point, then we have $\gamma'\neq 0$ and $\gamma'\gamma''=\gamma'\gamma'''=\cdots=\gamma'\gamma^{(p-1)}=0$. Therefore expanding the curve γ using Taylor development up to the order q at points $B=\gamma(t_o+\alpha)$ and $C=\gamma(t_o+\beta)$, we get

$$\left\|\overrightarrow{BC}\right\|^{q+1}=\left|(\beta-\alpha)^{q+1}\right|\left\|\gamma'\right\|^{q+1}=\left|\sum_{i=0}^{q+1}\binom{q+1}{i}(-1)^i\beta^{q+1-i}\alpha^i\right|\left\|\gamma'\right\|^{q+1}$$

$$\cong\left|(q+1)\alpha\beta\left(\beta^{q-1}+(-1)^q\alpha^{q-1}\right)\right|\left\|\gamma'\right\|^{q+1} \quad \cdots(3.1.4)$$

or

$$\left\|\overrightarrow{BC}\right\|^{q+1}=\left|(\beta-\alpha)^2(\beta-\alpha)^{q-1}\right|\left\|\gamma'\right\|^{q+1}\cong\left|\alpha\beta(\beta-\alpha)^{q-1}\right|\left\|\gamma'\right\|^{q+1} \quad \cdots(3.1.5)$$

And

$$\overrightarrow{AC}\times\overrightarrow{AB}=\begin{vmatrix} e_1 & e_2 & e_3 \\ \sum_{k=1}^{q}\frac{\beta^k x^k}{k!}+\delta_2\beta^q & \sum_{k=1}^{q}\frac{\beta^k y^k}{k!}+\delta_2\beta^q & \sum_{k=1}^{q}\frac{\beta^k z^k}{k!}+\delta_2\beta^q \\ \sum_{k=1}^{q}\frac{\alpha^k x^k}{k!}+\delta_1\alpha^q & \sum_{k=1}^{q}\frac{\alpha^k y^k}{k!}+\delta_1\alpha^q & \sum_{k=1}^{q}\frac{\alpha^k z^k}{k!}+\delta_1\alpha^q \end{vmatrix}$$

$$\cong\frac{\alpha\beta(\alpha^{q-1}-\beta^{q-1})}{q!}\left((y'z^{(q)}-z'y^{(q)})e_1+(z'x^{(q)}-x'z^{(q)})e_2+(x'y^{(q)}-y'x^{(q)})e_3\right)\cdots(3.1.6)$$

From equation **(3.1.5)** and **(3.1.6)** we get

$$\kappa_G(t)\cong\frac{\left|\left(\frac{\beta}{\alpha}\right)^{q-1}-1\right|\left|\gamma'(t)\times\gamma^{(q)}(t)\right|}{\left|\frac{\beta}{\alpha}-1\right|^{q-1}q!\left\|\gamma'(t)\right\|^{q+1}}$$

Since $\beta=\alpha\varepsilon$, which implies $\frac{\beta}{\alpha}\cong 0$, then

$$\kappa_G(t) \cong \frac{|\gamma'(t)\times\gamma^{(q)}(t)|}{q!\|\gamma'(t)\|^{q+1}}$$

Proof:

Using equations **(3.1.4)** and **(3.1.6)** we get

$$\kappa_G(t) \cong \frac{\left|\frac{\alpha\beta(\alpha^{q-1}-\beta^{q-1})}{q!}\right| |\gamma'(t)\times\gamma^{(q)}(t)|}{|\alpha\beta(q+1)(\beta^{q-1}+(-1)^q\alpha^{q-1})| \|\gamma'(t)\|^{q+1}}$$

Since q is an odd number, therefore $\kappa_G(t) \cong \dfrac{|\gamma'(t)\times\gamma^{(q)}(t)|}{(q+1)!\|\gamma'(t)\|^{q+1}}$.

Case III\ A is a Singular point:

Let $A=\gamma(t_o)$ be a singular point of order *p-1* and q the order of the first derivative not coplanar with $\gamma^{(p)}$, then using equation **(3.1.5)** we get

$$\|\overrightarrow{BC}\| = \left(\left(0+\cdots+\frac{\beta^p-\alpha^p}{p!}x^p+i.s\right)^2+\left(0+\cdots+\frac{\beta^p-\alpha^p}{p!}y^p+i.s\right)^2+\left(0+\cdots+\frac{\beta^p-\alpha^p}{p!}z^p+i.s\right)^2\right)^{\frac{1}{2}}$$

$$=\left|\frac{\beta^p-\alpha^p}{p!}\right|\left(x^{(p)2}+y^{(p)2}+z^{(p)2}\right)^{\frac{1}{2}}=\left|\frac{\beta^p-\alpha^p}{p!}\right|\|\gamma^{(p)}(t)\|$$

Thus

$$\|\overrightarrow{BC}\|^{\frac{q}{p}+1}=\|\overrightarrow{BC}\|^2\|\overrightarrow{BC}\|^{\frac{q}{p}-1}\cong\left|\frac{(\alpha^p\beta^p)(\beta^p-\alpha^p)^{\frac{q}{p}-1}}{(p!)^{\frac{q}{p}+1}}\right|\|\gamma^{(p)}(t)\|^{\frac{q}{p}+1}, \qquad \cdots(3.1.7)$$

$$\overrightarrow{AC}\times\overrightarrow{AB}\cong\frac{\alpha^p\beta^p(\alpha^{q-p}-\beta^{q-p})}{p!q!}\left((y^{(p)}z^{(q)}-z^{(p)}y^{(q)})e_1+(z^{(p)}x^{(q)}-x^{(p)}z^{(q)})e_2+(x^{(p)}y^{(q)}-y^{(p)}x^{(q)})e_3\right),$$

therefore

$$|\overrightarrow{AC}\times\overrightarrow{AB}|\cong\left|\frac{\alpha^p\beta^p(\alpha^{q-p}-\beta^{q-p})}{p!q!}\right|\|\gamma^{(p)}(t)\times\gamma^{(q)}(t)\| \qquad \cdots(3.1.8)$$

From equations (3.1.7) and (3.1.8) we have;

$$\kappa_G(t) \cong \frac{\left|\overrightarrow{AC} \times \overrightarrow{AB}\right|}{\left\|\overrightarrow{BC}\right\|^{\frac{q}{p}+1}} = \frac{\left|\frac{\alpha^p \beta^p \left(\alpha^{q-p} - \beta^{q-p}\right)}{p!q!}\right| \left\|\gamma^{(p)}(t) \times \gamma^{(q)}(t)\right\|}{\left|\frac{\left(\alpha^p \beta^p\right)\left(\beta^p - \alpha^p\right)^{\frac{q}{p}-1}}{(p!)^{\frac{q}{p}+1}}\right| \left\|\gamma^{(p)}(t)\right\|^{\frac{q}{p}+1}}$$

Since $\beta = \alpha\varepsilon$ which implies $\frac{\beta}{\alpha} \cong 0$ therefore

$$\kappa_G(t) \cong \frac{(p!)^{\frac{q}{p}} \left\|\gamma^{(p)}(t) \times \gamma^{(q)}(t)\right\|}{q! \left\|\gamma^{(p)}(t)\right\|^{\frac{q}{p}+1}}$$

Corollary 3.2

If $A = \gamma(t_o)$ is the biregular point of γ, then for any value of α and β, the general case of the generalized curvature of γ is given by

$$\kappa_G \cong \frac{\left\|\gamma'(t) \times \gamma''(t)\right\|}{c \cdot \left\|\gamma'(t)\right\|^3}$$

Corollary 3.3

If $A = \gamma(t_o)$ is the only regular point of γ and q is the order of the first derivative which is not collinear with γ', and $B = \gamma(t+\alpha)$, $C = \gamma(t+\beta)$ are two points infinitely close to the point $A = \gamma(t_o)$, then for any value of α and β, the generalize curvature of γ is given by:

$$\kappa_G(t) \cong \frac{\left|\gamma'(t) \times \gamma^{(q)}(t)\right|}{(q+1)! \left\|\gamma'(t)\right\|^{q+1}},$$ provided that q is an odd number.

Corollary3.4

Let $A = \gamma(t_o)$ be a singular point of order *p-1* and q the order of the first derivative not coplanar with $\gamma^{(p)}$, and $B = \gamma(t+\alpha)$, $C = \gamma(t+\beta)$ be two points infinitely close to the point $A = \gamma(t_o)$. Then for any value of α and β, the general curvature of γ is given by

$$\kappa_G(t) \cong \frac{(p!)^{\frac{q}{p}} \left\|\gamma^{(p)}(t) \times \gamma^{(q)}(t)\right\|}{\left(\frac{q}{p}+1\right) q! \left\|\gamma^{(p)}(t)\right\|^{\frac{q}{p}+1}},$$ provided that $\frac{q}{p}$ is an odd number.

Proof:

Since we have

$$\left\|\overrightarrow{BC}\right\|^{\frac{q}{p}+1} = \left|\left(\frac{\beta^p - \alpha^p}{p!}\right)^{\frac{q}{p}+1}\right| \left\|\gamma^{(p)}(t)\right\|^{\frac{q}{p}+1}$$

$$= \frac{1}{(p!)^{\frac{q}{p}+1}} \left| \sum_{i=0}^{\left[\frac{q}{p}+1\right]} C_i^{\left[\frac{q}{p}+1\right]} (-1)^i (\beta^p)^{\frac{q}{p}+1-i} (\alpha^p)^i \right| \left\|\gamma^{(p)}\right\|^{\frac{q}{p}+1} ,$$

thus

$$\left\|\overrightarrow{BC}\right\|^{\frac{q}{p}+1} \cong \left(\frac{q}{p}+1\right) \alpha^p \beta^p \left(\beta^{q-p} + (-1)^{\frac{q}{p}} \alpha^{q-p}\right) \cdot \frac{1}{(p\,!)^{\frac{q}{p}+1}} \left\|\gamma^{(p)}\right\|^{\frac{q}{p}+1} , \qquad \cdots(3.1.9)$$

therefore, using equations **(3.1.8)** and **(3.1.9)** we get

$$\kappa_G(t) \cong \frac{(p!)^{\frac{q}{p}} \left\|\gamma^{(p)}(t) \times \gamma^{(q)}(t)\right\|}{\left(\frac{q}{p}+1\right) q! \left\|\gamma^{(p)}(t)\right\|^{\frac{q}{p}+1}}$$ provided that $\frac{q}{p}$ is an odd number.

Theorem 3.5

Let γ be a standard curve of type C^∞ in $\mathbf{R}^3$ that admits at each point three vector derivatives not coplanar, denoted by $\gamma^{(p)}$, $\gamma^{(q)}$ and $\gamma^{(s)}$, where *p*, *q* and *s* are the smallest integers such that $\gamma^{(p)} \times \gamma^{(q)} \neq 0$, $\gamma^{(p)} \times \gamma^{(s)} \neq 0$ and $\gamma^{(p)} \times \gamma^{(q)} \neq 0$. Then the generalized curvature of γ is given by

$$^{o}\kappa_G = \frac{(p!)^{\frac{q}{p}} \left|\gamma^{(p)} \times \gamma^{(q)}\right|}{q! \left\|\gamma^{(p)}\right\|^{\frac{q}{p}+1}} .$$

Proof:

Let $B = \gamma(t+\alpha)$ and $C = \gamma(t+\alpha\varepsilon)$ where $\alpha, \varepsilon \cong 0$ be two points infinitely close to the point $A = \gamma(t)$. Then expand the curve γ using Taylor development up to the order *s* at each of the points $B = \gamma(t_o + \alpha)$ and $C = \gamma(t_o + \beta)$, we get:

$$\left\|\overrightarrow{BC}\right\| = \left(\left(0 + \cdots + \frac{\beta^p - \alpha^p}{p!} x^p + i.s\right)^2 + \left(0 + \cdots + \frac{\beta^p - \alpha^p}{p!} y^p + i.s\right)^2 + \left(0 + \cdots + \frac{\beta^p - \alpha^p}{p!} z^p + i.s\right)^2\right)^{\frac{1}{2}}$$

thus

$$\left\|\overrightarrow{BC}\right\|^{\frac{q}{p}+1} = \left\|\overrightarrow{BC}\right\|^{2}\left\|\overrightarrow{BC}\right\|^{\frac{q}{p}-1} \cong \left|\frac{\left(\alpha^{p}\beta^{p}\right)\left(\beta^{p}-\alpha^{p}\right)^{\frac{q}{p}-1}}{(p!)^{\frac{q}{p}+1}}\right| \left\|\gamma^{(p)}(t)\right\|^{\frac{q}{p}+1} \qquad \cdots(3.2.1)$$

therefore

$$\overrightarrow{AC}\times\overrightarrow{AB} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \sum_{k=1}^{s}\frac{\beta^k x^k}{k!}+\delta_2\beta^s & \sum_{k=1}^{s}\frac{\beta^k y^k}{k!}+\delta_2\beta^s & \sum_{k=1}^{s}\frac{\beta^k z^k}{k!}+\delta_2\beta^s \\ \sum_{k=1}^{s}\frac{\alpha^k x^k}{k!}+\delta_1\alpha^s & \sum_{k=1}^{s}\frac{\alpha^k y^k}{k!}+\delta_1\alpha^s & \sum_{k=1}^{s}\frac{\alpha^k z^k}{k!}+\delta_1\alpha^s \end{vmatrix} =$$

$$\left(\frac{\alpha^q\beta^q(\alpha^{s-q}-\beta^{s-q})}{s!q!}(y^q z^s - y^s z^q)\right)e_1 +$$
$$\left(\frac{\alpha^p\beta^p(\alpha^{s-p}-\beta^{s-p})}{s!p!}(x^p z^s - x^s z^s)\right)e_2 +$$
$$\left(\frac{\alpha^p\beta^p(\alpha^{q-p}-\beta^{q-p})}{q!p!}(x^q y^p - x^p z^q)\right)e_3 \quad .$$

Thus

$$\kappa_G(t) \cong \frac{\left|\overrightarrow{AC}\times\overrightarrow{AB}\right|}{\left\|\overrightarrow{BC}\right\|^{\frac{q}{p}+1}} = \left(\frac{(p!)^{\frac{q}{p}+1}\beta^{q-p}\left(\alpha^{s-q}-\beta^{s-q}\right)}{s!q!\left(\frac{\beta}{\alpha}-1\right)^{\frac{q}{p}-1}\left\|\gamma^{(p)}(t)\right\|^{\frac{q}{p}+1}}\left(y^q z^s - y^s z^q\right)\right)e_1 +$$

$$\left(\frac{(p!)^{\frac{q}{p}}\alpha^{s-p}\left(1-\left(\frac{\beta}{\alpha}\right)^{s-p}\right)}{s!\left(\alpha^p\right)^{\frac{q}{p}-1}\left(\left(\frac{\beta}{\alpha}\right)^{p}-1\right)^{\frac{q}{p}-1}\left\|\gamma^{(p)}(t)\right\|^{\frac{q}{p}+1}}\left(x^p z^s - x^s z^s\right)\right)e_2 +$$

$$\left(\frac{(p!)^{\frac{q}{p}}\left(1-\left(\frac{\beta}{\alpha}\right)^{q-p}\right)}{q!\left(\left(\frac{\beta}{\alpha}\right)^{p}-1\right)^{\frac{q}{p}-1}\left\|\gamma^{(p)}(t)\right\|^{\frac{q}{p}+1}}\left(x^p y^q - x^q y^p\right)\right)e_3$$

Since α and β are arbitrary chosen so we may assume that $\frac{\beta}{\alpha}$ is infinitesimal, thus we get

$$^{o}\kappa_G = \frac{(p!)^{\frac{q}{p}}\left|x^{(p)}y^{(q)} - x^{(q)}y^{(p)}\right|}{q!\left\|\gamma^{(p)}\right\|^{\frac{q}{p}+1}} = \frac{(p!)^{\frac{q}{p}}\left|\gamma^{(p)} \times \gamma^{(q)}\right|}{q!\left\|\gamma^{(p)}\right\|^{\frac{q}{p}+1}}$$

Theorem 3.6

Let γ be a standard curve of type C^{∞} in $\mathbf{R}^3$ which admits at each point two vector derivatives not coplanar denoted by $\gamma^{(p)}$ and $\gamma^{(q)}$, where p and q are the smallest integers such that $\gamma^{(p)} \times \gamma^{(q)} \neq \mathbf{0}$. Then the generalized curvature of γ is equal to the shadow $^{o}\left|\frac{\varepsilon_2}{\varepsilon_1^{\frac{q}{p}-1}}\right|$.

Proof:

Form the Decomposition Theorem we have

$$\varepsilon_1\overrightarrow{V_1} + \varepsilon_1\varepsilon_2\overrightarrow{V_2} + \varepsilon_1\varepsilon_2\varepsilon_3\overrightarrow{V_3} = \frac{\gamma^{(p)}(t_o)}{p!}\delta^p + \cdots + \frac{\gamma^{(q)}(t_o)}{q!}\delta^q + \cdots + \frac{\gamma^{(s)}(t_o)}{s!}\delta^s + \eta\delta^s \quad \cdots(3.3.1)$$

Where $\eta \cong \mathbf{0}$. Dividing equation **(3.3.1)** by ε_1, we get

$$\overrightarrow{V_1} + \varepsilon_2\overrightarrow{V_2} + \varepsilon_2\varepsilon_3\overrightarrow{V_3} = \frac{\gamma^{(p)}(t_o)}{p!}\frac{\delta^p}{\varepsilon_1} + \cdots + \frac{\gamma^{(q)}(t_o)}{q!}\frac{\delta^q}{\varepsilon_1} + \cdots + \frac{\gamma^{(s)}(t_o)}{s!}\frac{\delta^s}{\varepsilon_1} + \eta\frac{\delta^s}{\varepsilon_1} \quad \cdots(3.3.2)$$

thus

$$\overrightarrow{V_1} \cong \frac{\gamma^{(p)}(t_o)}{p!}\frac{\delta^p}{\varepsilon_1} = \frac{\gamma^{(p)}(t_o)}{p!}\frac{p!}{\left\|\gamma^p\right\|} = \frac{\gamma^{(p)}(t_o)}{\left\|\gamma^p\right\|} \quad \cdots(3.3.3)$$

The cross product of V_1 by equation **(3.3.1)** implies

$$\varepsilon_1\varepsilon_2\overrightarrow{V_1}\times\overrightarrow{V_2} + \varepsilon_1\varepsilon_2\varepsilon_3\times\overrightarrow{V_1}\times\overrightarrow{V_3} \cong \frac{\gamma^{(p)}\times\gamma^{(q)}}{q!\left\|\gamma^{(p)}\right\|}\delta^q + \cdots + \frac{\gamma^{(p)}\times\gamma^{(s)}}{s!\left\|\gamma^{(p)}\right\|}\delta^s + \eta\frac{\delta^s}{\varepsilon_1} \quad \cdots(3.3.3)$$

Since each of V_1 and V_2 are orthonormal unit vectors then equation (3.3.3) becomes

$$\varepsilon_1\varepsilon_2 \cong \frac{\gamma^{(p)}\times\gamma^{(q)}}{q!\left\|\gamma^{(p)}\right\|}\delta^q, \qquad \text{that is} \qquad \left|\varepsilon_1\varepsilon_2\right| \cong \frac{\left\|\gamma^{(p)}\times\gamma^{(q)}\right\|}{q!\left\|\gamma^{(p)}\right\|}\left|\delta^q\right|$$

Moreover from equation (2.4.3) of the proof of Decomposition Version of Generalized Curvature Theorem, we have $\left|\varepsilon_1\right| \cong \frac{\left\|\gamma^{(p)}\right\|}{p!}\left|\delta^p\right|$

Now rising $|\varepsilon_1|$ to the power *q* and $|\varepsilon_1\varepsilon_2|$ to the power *p*, we have

$$\left|\frac{\varepsilon_2}{\varepsilon_1^{\frac{q}{p}-1}}\right| \cong \frac{(p!)^{\frac{q}{p}}\left\|\gamma^{(p)}\times\gamma^{(q)}\right\|}{q!\left\|\gamma^{(p)}\right\|^{\frac{q}{p}+1}} = {}^{o}\kappa_G \ ,$$

thus

$${}^{o}\kappa_G = \left|\frac{\varepsilon_2}{\varepsilon_1^{\frac{q}{p}-1}}\right|$$

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