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The Lower Bounds of Eight and Fourth Blocking Sets and Existence of Minimal Blocking Sets

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المستخلص

في هذا البحث حصلنا على نتيجتين رئيسيتين متعلقتين بحجم المجموعة القالبية من النمط-8والنمط-4 في المستوي PG(2,16). وتمثلت النتيجتان في إيجادنا قوسا تاما جديدا – (129,9) لم يسبق الحصول عليه في البحوث الحديثة، وإثباتنا أن القوس التام -(k,13)موجود في المستوي PG(2,16) عندما k≤197. وقمنا بتصنيف المجاميع القالبية الاصغرية ذات الحجم 8 في المستوي الاسقاطي PG(2,4) ، وأثبتنا أن المجموعة الاصغرية ذات الحجم 8 هي من النوع Rédei –type وأعطينا بعض خواص المجموعة القالبية الاصغرية ذات الحجم 10 في المستوي الاسقاطي PG(2,5) ، وحصلنا على مثال للمجموعة القالبية الاصغرية ذات الحجم 10 ذات قاطع رباعي على الأكثر في المستوي الاسقاطي PG(2,5) وأثبتنا أن المجموعة الاصغرية ذات الحجم 10 هي من النوع Rédei –type

ABSTRACT

This paper contains two main results relating to the size of eight and fourth blocking set in PG(2,16). First gives new example for (129,9)-complete arc. The second result we prove that there exists (k,13)-complete arc in PG(2,16), k≤197. We classify the minimal blocking sets of size eight in PG(2,4).We show that Rédei –type minimal blocking sets of size eight exist in PG(2, 4). Also we classify the minimal blocking sets of size ten in PG(2, 5), We obtain an example of a minimal blocking set of size ten with at most 4-secants.We show that Rédei –type minimal blocking sets of size ten exists in PG(2, 5).

ملاحظة:البحث مستل من الأطروحة

1.1 Introduction:

A (k ,n)-arc K in PG(2,q) is a set of k points such that there is some n but no n+1 of them are collinear. A (k ,n)-arc K is complete if there is no (k+1,n)-arc containing it. The maximum value of k which a (k ,n)-arc K exist in PG(2,q) will be denoted by $m(n)_{2,q}$[6].

A t-fold blocking set B in a projective plane, is a set of points such that each line contains at least t points of B and some line contains exactly t points of B [1]. For t=1,a1-fold blocking set is called a blocking set. A trivial blocking set B is a blocking set containing a line of PG(2,q). A t- blocking set is called minimal (irreducible)when no proper subset of it is a t- blocking set [12]. For t=2,3,4,…then t- blocking set is called respectively double blocking set, triple blocking set , fourth blocking set…etc. (k ,n)-arcs and t- blocking sets are in fact just complements of each other in a projective plane , with $n + t = q + 1$.

Richardson was the first one to look at larger planes [11]. He showed that the minimal size of a blocking set in PG(2 ,3) is 6,and noted that Baer subplanes are examples of blocking sets of size $q+\sqrt{q}+1$ in projective planes of square order. After that things were quiet for 13 years until Di paola[4] introduced the idea of a projective triangle, which gives an example of a blocking set of size 3(q+1)/2 in Desargusian planes of odd order. That projective triangles exist in these planes was shown by Bruen , who also obtained the general lower bound $q+\sqrt{q}+1$ for the size of a blocking set in arbitrary projective plane of odd order q.

Further results obtained by Bruen [3], giving the upper bound $q\sqrt{q}+1$ for a minimal blocking set in any projective plane of order q, and make the connection with Re`dei ′s work on lacunary polynomials[ 10 ]. The fundamental results are for the structure of blocking sets however was only realized much later and in this course the emphasis will be to explain in some detail the recent developments and the connection between Re`dei ′s work on lacunary polynomials and small blocking sets and multiple blocking sets in Desargusian projective planes.

1.2 The projective plane PG(2,16):

Let $f(x)=x^3+x^2+x+\lambda$ be a monic polynomial over GF(16) then companion matrix of f(x)

$$T=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & 1 & 1 \end{bmatrix}$$

is cyclic projectivety on PG(2,16).Note that in PG(2,16) ,$\{\lambda=-\lambda\}$

Let π =Gf(16)={0,1, λ^i :i $\in N_{14}$: λ^{15}=1}
We write the elements of π as 1,2,3…,16 instead of 0,1, $\lambda,\ldots,\lambda^{14}$, respectively. So the cyclic projectivety becomes:

$$T=\begin{bmatrix}1 & 2 & 1\\ 1 & 1 & 2\\ 3 & 2 & 2\end{bmatrix}$$

The number of point in the PG(2,16) has 273 points and 273lines and every line passes throw17 points.

Let p_0 be the point U_0 =(2,1,1)then $P_i=P_0T^i$, i=0,...,272,are the 273 points of PG(2,16). See [8,Table(1)]

Let L_1 be the line which contains the points 0,1,4,16,26,57,64,91,93,99,104,123,143,205,219,228,256, then $L_i=L_1T^{i-1}$,i=1,...,273, are the lines of PG(2,16), the 273 lines L_i are given by the rows in [8,Table(2)].

2.1 Eight blocking sets in PG(2, 16)

The object of this section is to obtain good lower bounds for the size of eight blocking sets in PG(2, q),q is square integer.

Theorem(2.1.1) (q>9,q is a square):

Let B be an eight blocking set in PG(2, q) , q is square such that through each of its points there are$\sqrt{q}$+1 lines, each lines contains at least $\sqrt{q}$+8 points of B and forming a dual Baer subline .Then

(1) For q>64 , B has at least 8q+2$\sqrt{q}$+8 points.

(2) For q=16, B has at least 8q+$\sqrt{q}$+10 points.

Proof. (1)Call the lines meeting B in $\sqrt{q}$+8 or more points long lines . If two long lines meet out side of B , then B has at least 2($\sqrt{q}$+8)+8(q-1)= 8q+2$\sqrt{q}$+8 points and the desired bound is obtained . Hence $|B|\geq 8q+2\sqrt{q}+8$. So to assume that two long lines meet in B .Take l , a long line ,and p ,a point of B not on l. Then the long lines through p contain a dual Baer subline and meet l in a Baer subline. Let Q be a point on this Baer subline. Consider long lines through a point on an 8-secant to Q. These meet l in another Baer subline not containing Q. Two Baer sublines meet in at most two points and so l has at least 2$\sqrt{q}$ points . Since l was arbitrary every long line has at least 2$\sqrt{q}$ points and it follows that B has at least ($\sqrt{q}$+1)(2$\sqrt{q}$-1)+1+7(q-$\sqrt{q}$)=9q-6$\sqrt{q}$ points.
Since $9q-6\sqrt{q}\geq 8q+2\sqrt{q}+8$ so that $|B|\geq 8q+2\sqrt{q}+8$ points.

Proof. (2) If two long lines meet out side of B , then B has at least $2(\sqrt{q}+8\ )+8(q-1)= 8q+2\sqrt{q}+8$ points. Hence $|B|\geq 8q+2\sqrt{q}+8$.

Let $p\in B$, through B , since there are $\sqrt{q}+1$ long lines through p. B has at least $(\sqrt{q}+1)(\sqrt{q}+7)+1+7(q+1-(\sqrt{q}+1))=8q+\sqrt{q}+8$ points. Now $|B|\geq 140$. If this bound is a chafed then (k ,9)-arc has k=133 and that impossible .See Table(3)from [2]**.** If $|B|=8q+\sqrt{q}+9$ then k=132, that impossible. Since $k\leq 131$, hence $|B|\geq 8q+\sqrt{q}+10$.

Table (3)
The size of the largest (k ,n)-arc in PG(2,q) for small q

n \ q	3	4	5	7	8	9	11	13	16	17	19
2	4	6	6	8	10	10	12	14	18	18	20
3		9	11	15	15	17	21	23	28…33	28…35	31…39
4			16	22	28	28	32…34	38…40	52	48…52	52…58
5				29	33	37	43…45	49…53	65	61…69	68…77
6				36	42	48	56	64…66	78…82	78…86	86…96
7					49	55	67	79	93…97	94…103	105…115
8						65	77…78	92	120	114…120	124…134
9							89…90	105	128…131	137	147…153
10							100…102	118…119	142…148	154	172
11								132…133	159…164	166…171	191
12								145…147	180…181	182…189	204…210
13									195…199	204…207	225…230
14									210…214	221…225	242…250
15									231	239…243	262…271
16										256…261	285…290
17											305…311
18											324…330

Corollary (2.1.2):

There exists (129,9)-arc in PG(2,16).

Proof. Finding a maximum(k,9)-arc is equivalent to finding the minimum eight blocking sets by considering complements .

Theorem(2.1.1)gives lower Bound for eight blocking set with $8q+2\sqrt{q}+8$, if two lines with $\sqrt{q}+8$ points intersect outside of the eight blocking set. Eight blocking set must have at least 144 points there were eight blocking sets exactly 144 points, and equivalently a (129,9)-arc does exist Example(2.1.4). Hence k=129 is a new sharp upper bound for (k,9)-arc. See Table(3)**.**

2.1.3 The value of $m(n)_{2,q}$

In this section example of large (k ,n)-arcs in PG(2,16) are given. Improvements on the upper bounds of $m(n)_{2,16}$ obtained from Corollary (2.1.2) are made . Example(2.1.4) constructed by taking random subsets of the internal points of a conic.

Example(2.1.4) : The set of the following points

$\{(1,0,0),(0,1,0),(0,0,1),(1,1,1),(1,\lambda^6,\lambda^9),(1,\lambda^8,\lambda^7),(1,\lambda^{10},\lambda^5),(1,\lambda^5,\lambda^{10})$
$,(1,\lambda,\lambda^{14}),(1,\lambda^7,\lambda^8),(1,\lambda^{12},\lambda^3),(1,\lambda^2,\lambda^{13}),(1,\lambda^3,\lambda^{12}),(1,\lambda^{13},\lambda^2)$
$,(1,\lambda^{14},\lambda),(1,\lambda^9,\lambda^6),(1,\lambda^4,\lambda^{11}),(1,\lambda^{11},\lambda^4),(1,\lambda^8,\lambda^6),(1,\lambda^{14},\lambda^{14})$
$,(1,\lambda^7,\lambda^2),(1,\lambda^9,\lambda),(1,\lambda^3,\lambda^8),(1,\lambda^{11},\lambda^5),(1,\lambda^{10},\lambda^{13}),(1,\lambda^5,\lambda^9)$
$,(1,0,\lambda^7),(0,1,\lambda^8),(1,\lambda^{13},\lambda^4),(1,\lambda^6,0),(1,1,\lambda^{10}),(1,\lambda^{14},\lambda^{13}),$
$(1,\lambda^{13},1),(1,\lambda^3,\lambda^{14}),(1,\lambda^8,\lambda^9),(1,\lambda^6,\lambda^3),(1,\lambda,\lambda^8),(1,\lambda^{10},\lambda^6),(1,\lambda^4,\lambda^{12}),$
$(1,\lambda^9,\lambda^7),(1,1,\lambda),(1,\lambda^7,\lambda^4),(1,1,\lambda^3),(0,1,\lambda^9),(1,\lambda,1),(1,0,\lambda^{13})$
$,(1,\lambda^2,\lambda^7),(1,\lambda^{12},1),(1,\lambda^9,\lambda^{11}),(1,\lambda,\lambda^5),(1,\lambda^{11},\lambda^{12}),(1,\lambda^2,\lambda),(1,\lambda^5,\lambda^{11})$
$,(1,\lambda^{12},0),(0,1,\lambda^7),(1,\lambda^{10},12),(1,\lambda^5,\lambda^5),(1,\lambda^4,\lambda^2),(1,\lambda^7,\lambda^3),$
$(1,\lambda^4,\lambda^9),(1,1,\lambda^{12}),(1,\lambda^{13},\lambda),(1,0,\lambda^5),(0,1,\lambda^4),(1,\lambda,\lambda^{11}),(1,\lambda^{12},\lambda^{12}),$
$(1,\lambda^8,\lambda^4),(1,\lambda^6,\lambda^7),(1,\lambda^{14},\lambda^9),(1,\lambda^{12},\lambda^2),(1,\lambda^4,0),(1,\lambda^{11},\lambda^{13}),$
$(1,\lambda^2,\lambda^6),(1,\lambda^8,1),(1,\lambda^{10},0),(1,\lambda^7,\lambda^5),(1,0,\lambda^2),(1,\lambda^2,\lambda^8),$
$(1,\lambda^6,\lambda^{12}),(1,\lambda^{11},\lambda^{11}),(1,0,\lambda^9),(1,\lambda^5,\lambda^6),(0,1,\lambda^{13}),(1,\lambda^2,\lambda^{14}),$
$(1,\lambda^5,\lambda),(1,1,\lambda^4),(1,\lambda^{12},\lambda^4),(1,\lambda^6,\lambda^8),(0,1,\lambda^6),(1,\lambda^8,\lambda^3),$
$(1,\lambda,\lambda^9),(1,\lambda^4,\lambda^7),(1,\lambda^{14},\lambda^{10}),(1,0,\lambda^{14}),(1,\lambda^9,\lambda^{14}),(1,\lambda^2,\lambda^2),$
$(1,\lambda^{13},\lambda^{10}),(1,\lambda^7,\lambda^{13}),(1,1,\lambda^5),(1,\lambda^6,\lambda^{13}),(1,\lambda^8,\lambda^{12}),(1,\lambda,\lambda^{10}),$
$(1,\lambda^{13},\lambda^6),(1,\lambda^{10},\lambda^{11}),(1,\lambda^8,\lambda),(1,\lambda^5,\lambda^3),(1,\lambda^4,\lambda^3),$
$(1,\lambda^{12},\lambda^{11}),(1,\lambda^6,1),(1,\lambda^9,\lambda^5),(1,\lambda^{10},\lambda^4),(1,\lambda^7,1),(1,\lambda^3,\lambda^8),$
$(1,\lambda^{14},\lambda^8),(1,\lambda^3,\lambda^7),(1,\lambda^3,\lambda^{10}),(1,\lambda^4,\lambda^6),(1,1,\lambda^9),(1,\lambda,\lambda^{13}),$
$(1,\lambda^7,0),(0,1,\lambda^2),(1,\lambda^3,\lambda^3),(1,\lambda^{11},\lambda^{14}),(1,\lambda^9,\lambda^2),(1,\lambda^3,\lambda)$
$,(1,\lambda^{11},\lambda^{10}),(1,\lambda^5,0),(1,\lambda^{14},\lambda^4),(1,0,\lambda^{11})\}$. Forms a (129,9)-arc in PG(2,16) with secant distribution

$T_0=8, T_1=9, T_2=0, T_3=0, T_4=0, T_5=0, T_6=0, T_7=0, T_8=120$ and $T_9=136$.

2.2 Fourth blocking sets in PG(2, 16)

The object of this section is to obtain good lower bounds for the size of a fourth blocking sets in PG(2, q) , q is square.

Theorem (2.2.1) (q>9, q is a square)

Let B be a fourth blocking set in PG(2, q) , q is square, such that through each of its points there are $\sqrt{q}+1$ lines, each containing at least $\sqrt{q}+4$ points of B and forming a dual Baer subline . Then B has at least $4q+2\sqrt{q}+4$ points.

Proof. Call the lines meeting B in $\sqrt{q}+4$ or more points long lines .If two long lines meet out side of B ,then B has at least

$2(\sqrt{q}+4)+4(q-1)= 4q+2\sqrt{q}+4$ points and the desired bound is obtained .So assume that two long lines meet in B . let l be a long line and p a point of B not on l.Then the long lines through p contain a dual Baer subline and meet l in a Baer subline. Let Q be a point on this Baer subline. Consider long lines through a point on a 4-secant to Q. These meet l in another Baer subline not containing Q. Two Baer subline meets in at most two points and so l has at least $2\sqrt{q}$ points . Since l was arbitrary every long line has at least $2\sqrt{q}$ points and it follows that B has at least $(\sqrt{q}+1)(2\sqrt{q}-1)+1+3(q-\sqrt{q})=5q-2\sqrt{q}$ points.

For q>16, q square $5q-2\sqrt{q}\geq 4q+2\sqrt{q}+4$. If q=16 then $5q-2\sqrt{q}=72$ and (k,13) has 201 points and that is impossible ,see Table(3). Therefore

$|B|\geq 4q+2\sqrt{q}+4$.

Corollary (2.2.2):

There exists (k ,13)-arc in PG(2,16), $k \leq 197$

Proof. Finding a maximum(k,13)-arc is equivalent to finding the minimum fourth blocking set by considering complements . Theorem (2.2.1)gives lower bound for fourth blocking set with $4q+2\sqrt{q}+4$. Fourth blocking set must have at least 76 points, so since n=q+1-t, then (k,13)-arcs have $k \leq 197$.

3.1 On Blocking sets:

In this section we have given the following information on the structure of such blocking sets.

Definition (3.1.1) (unital):[3]

Points, that every line $q\sqrt{q}+1$ A unital in PG(2,q) is a set U of

$\sqrt{q}+1$ joining two points of U intersects U in precisely

points.

again straight forward counting gives that all other lines of the plane there is intersect U in precisely one point, and in fact at each point of U a unique tangent. So a unital is a minimal blocking set. In fact it turns out to be the largest one.

Theorem(3.1.2) : [3]

let B be a minimal blocking set in PG(2, q). Then $|B|\leq q\sqrt{q}+1$ with equality if and only if B is a unital in PG(2, q) ,q is square .

Theorem (3.1.3) :[7]

In PG(2,q),q square , $q\geq 25$ or q=9, there is no minimal blocking set of size $q\sqrt{q}$.

Theorem(3.1.4) :[7]

For q square , $q \geq 16$, there is no minimal blocking k-set B $\leq k \leq q+\sqrt{q}+1$ in PG(2,q) with $q+2\sqrt{q}+1$

Theorem(3.1.5) :[7]

In a Desargusian plane of order at least 4 there exists a blocking set of order k if $2q-1 \leq k \leq 3q-3$.

3.2 Minimal Blocking sets in PG(2,4):

From now on, let B be a minimal blocking set of size eight in PG(2, 4), since B is non-trivial a line l intersect B in at most four points.

Lemma (3.2.1): There's at most two 4-secants through any point of B.

Proof. Every two 4-secant to B are intersect in a point on B.If two 4-secants intersect in $p \notin B$ then $|B| \geq 2 * 4+3=11$,which is impossible , Assume there is a three 4-secant through a point $p \in B$, then $|B| \geq 1+3*3=10$ and that is impossible. So through every point of B there is at most two 4-secants.

Lemma (3.2.2): If B has no 4- secants, then B has at least one secant with at least three points.

Proof. Suppose there are only 1-,2-secants,let the number of them be denoted by a and b. Then the following equations must hold by standard counting arguments.

a+b =21 ...(1)
a+2b =40 ...(2)
2b=56 ... (3)

From equation(3), we get b=28 which is impossible.

Lemma (3.2.3): If B has no 3-secant, then B has at least one 4-secant.

Proof. Suppose there are only 1-,2-,and 4- secants. let the number of them be denoted by a ,b, d recp. Then the following equations must hold by standard counting arguments.

a+ b+ d =21 .. .(1)
a+2b+4d =40 ...(2)
2b+12d=56 ... (3)

From these equations, we get d=3 .

Theorem(3.2.4) : Let B be a non-trivial blocking set . Let the number of 1-, 2-, 3- and 4-secants be denoted by a, b, c, d resp . Then we have one of the following possibilities:

a	B	c	d	Possibilities
8	10	0	3	(i)
9	7	3	2	(ii)
10	4	6	1	(iii)
11	1	9	0	(iv)

Proof. The standard counting arguments give:

$a + b + c + d = 21$ ... (1)

$a + 2b + 3c + 4d = 40$... (2)

$2b + 6c + 12d = 56$ ... (3)

From these we can deduce

$a = 11 - d$;

$b = 1 + 3d$;

$c = 9 - 3d$;

Since $c \geq 0$, then $0 \leq d \leq 3$.

We first show that first and fourth solution of Theorem (3.2.4) are not possible:

Theorem(3.2.5):

The first solution (8,10,0,3) and fourth solution (11,1,9,0) of Theorem (3.2.4) do not exists.

Proof. (i)let B be a blocking set having the solution (8,10,0,3), and assume l_1, l_2, l_3 be the three 4-secants of B: If $l_1 \cap l_2 \cap l_3 = \{p\}$. Then p must be in B, and that contradicts Lemma(3.2.1). Now if l_1, l_2, l_3 are triangular , so $|B| \geq 9$ and that is impossible . So solution (8,10,0,3)does not exist.

(iii) Let B be a blocking set having the solution (11,1,9,0).Since $c>0$,let ℓ be a 3-secant. Now any two 3-secant must be intersect in a point of B. Since if two3-secant intersect in a point $p \notin B$, then $|B| \geq 2*3+3*1=9$ which is impossible. On every $p \in B$ there are at most three 3-secants passing through p.

Now since $T_3=9$ then the remaining eight 3-secants pass through the three points of $\ell \cap B$, So we have a point of $\ell \cap B$ with at least four 3-secants, and that is impossible. Hence (11,1,9,0) does not exist.

The following lemma gives crucial information on the structure of such a blocking set . This lemma was proved by Ga´cs [5] using the Re´dei-polynomial [10]. It will enable us to eliminate the existence of such minimal blocking sets.

Lemma(3.2.6) :Ga´cs [5]. In PG(2,q)let B be a minimal blocking set of size
q+ k, and suppose there is a line l intersecting B in exactly k - 1 points. Then there is a point O$\notin$ B such that every line joining O to a point of l\ B contains two points of B. Hence k$\geq$ (q+3)/2.

The only possibility for a minimal blocking set of size eight in PG(2,4) that remains is a blocking set containing a 4-secant ; in other words a blocking set of Rédei –type.

Theorem(3.2.7) . There is a minimal blocking set of size eight of Rédei –type in PG(2,4) .

Proof. Let(x, y , z) denote the coordinates of a projective point. Let l be a 3-secant to B. Let l be the line at infinity (z=0) of the corresponding affine plane, and let {P1, P2}= l\ B. By Lemma (4.2.6) , there is an affine point O$\notin$ B for which the lines OPi, i = 1,2, are bisecants . These lines contain four affine points of B. Let U be the 5th affine point of B\l. Since the points Pi only lie on bisecant and three tangents, the lines UPi are tangents for i =1, 2.
Furthermore, the line OU is a line passing through a point of B$\cap$ l.

Let P_1 =(1,0,0) , P_2 =(0,1,0) , Assume OU passing through (1,1,0). Since no three of {P_1,P_2,O,U}are collinear we can consider O=(0,0,1) ,U=(1,1,1).

Consider now the affine plane PG(2,4)\ l. Let B'= B\(l $\cup$ {U}). Then two points of B' lie on X = 0 , two on Y = 0.Since these are the lines OPi , i =1,2 . Moreover , on every horizontal line Y= k , vertical line X = k, and on every line there is one point of B ,in particular on line y=1,x=1,y=x which all passing through U there is no point of B', Let the points of AG(2,4) be .

(0,0) ,(0,1) , (0 ,w) , (0,w^2)
(1,0) , (1,1) , (1 ,w) , (1,w^2)
(w,0) ,(w,1) , (w,w) ,(w,w^2)
(w^2,0) ,(w^2,1) ,(w^2,w) ,(w^2,w^2)

On OP_1 ;Y =0, the remaining two points which are not belonging to any line through U are l_1={(w,0),(w^2,0)} .

On OP_2 ;X =0, the remaining two points which are not belonging to any line through U are l_2={(0 ,w) , (0,w^2)}. Chosen the point (0 ,w), (0,w^2), on x=0 does not eliminate any points of l_1 also chosen (w, 0) ,(w^2,0) does not eliminate any points of l_2 ; in B.So the set B$\cap$ l $\cup$ {(w,0,1),(w^2 ,0,1),(0,w,1),(0,w^2,1),(1,1,1)}=
{(1,1,0),(w,1,0),(w^2,1,0), (w,0,1),(w^2 ,0,1),(0,w,1),(0,w^2,1),(1,1,1)} form a minimal blocking set of Rédei –type.

3.3 Minimal Blocking sets in PG(2 , 5):

The following lemmas give the properties of minimal blocking sets of size ten.

Lemma(3.3.1): Every blocking set of size ten in PG(2,5) has at least four points on a line.

Proof. Suppose there are only 1-,2-, and 3-secants.let the number of them be denoted by a, b , c, resp. Then the following equations must hold by standard counting arguments.

a+b+c =31 ...(1)

a+2b+3c =60 ...(2)

2b+6c=90 ... (3)

From these equations, we get b= -3 which is impossible.

Lemma (3.3.2): There are at most three 4-secant through any point of B.

Proof. Every two 4-secants to B are intersected in a point on B, if two 4-secant intersect in p $\not\in$ B then $|B|\geq 2 * 4+4=12$, which is impossible , New assume there are four 4-secants through a point p $\in$ B, then $|B|\geq 3*4+1=13$ and that is impossible. So through every point of B there are at most three 4-secants.

Lemma(3.3.3): There are no minimal blocking sets of size ten with 4-secant but no 3-secant .

Proof. Suppose there are only 1-,2-,and 4- secants. Let the numbers of them be denoted by a,b,d,resp .Then the following equations must hold by standard counting arguments.

a+ b+ d =31 .. .(1)

a+2b+4d =60 ...(2)

2b+12d=90 ... (3)

From these equations we get 3d=16 which is not possible for 3 does not divide 16 .

Lemma (3.3.4): If B has no 2-secant, then B has at least one 4-secant

Proof. Suppose there are only 1-,3-,and 4- secants. Let the number of them be denoted by a, c ,d .Then the following equations must hold by standard counting arguments.

a+ c+ d =31 .. .(1)

a+3c+4d =60 ...(2)

6c+12d=90 ... (3)

From these equations we get d=1.

It is easy to prove.

Lemma(3.3.5) : Let l_1 be a 4-secant to B and l_2 be a 3-secant to B then $l_1 \cap l_2$ be a point in B.

Lemma(3.3.6) : Let l be a 4-secant to B then through any point of $l \cap B$ there is at most three 3-secant.

Proof. Let p be a point of $l \cap B$ and assume there are four 3-secant through p,then $|B| \geq 4+2*4=12$ which contradict the size of B.

Theorem (3.3.7): Let B have at most four points on a line. Let the number of 1-, 2-, 3- and 4-secants be denoted by a, b, c, d resp. Then these numbers satisfy one of the following possibilities:

a	b	c	d	Possibilities
13	12	1	5	(i)
14	9	4	4	(ii)
15	6	7	3	(iii)
16	3	10	2	(iv)
17	0	13	1	(v)

Proof. The standard counting arguments give:

$a + b + c + d = 31$ ... (1)

$a + 2b + 3c + 4d = 60$ ... (2)

$2b + 6c + 12d = 90$ ... (3)

From these we can deduce

$a = 18 - d$;

$b = -3 + 3d$;

$c = 16 - 3d$;

Since $c \geq 0$, we get $d \leq 5$.

Theorem(3.3.8): The solution (17,0,13,1)of Theorem(3.3.7)does not exist.

Proof. Let l be a 4-secant.Since there are thirteen 3-secants, and since every 3-secant must intersect the 4-secant l in a point in B, so we have a point p in B Through which pass at least four 3-secants, and that contradicts to Lemma (3.3.6).

3.3. 9 Minimal blocking sets of size ten with at most 4-secants:

We find an example of minimal blocking sets of size ten with ten points.

Example (3.3. 10) :In PG(2,5) the set of the points{(1,2,0),(1,-1,0),(0,1,-1), (1,-2,0) ,(0,1,-2),(1,1,2),(1 ,1,0),(1,1,1),(1,0,-1),(1,0,-2)} is minimal blocking set with $T_1=14, T_2=9, T_3=4, T_4=4, T_5=0$.

3.3.11 Minimal blocking sets of size ten with 5-secants:

The following theorems prove that the existence of minimal blocking sets of size ten, $T_5>0$, $T_4\neq 0$.

Theorem(3.3.12): Let B have at most 5 points on a line. Let the numbers of 1-, 2-, 3-,4- and 5-secants be denoted by a, b, c, d ,e resp . Then these numbers satisfy one of the following possibilities:

a	b	c	d	e	Possibilities
11	16	1	1	2	(i)
12	13	4	0	2	(ii)
12	14	1	3	1	(iii)
13	11	4	2	1	(iv)
14	8	7	1	1	(v)
15	5	10	0	1	(vi)

Proof. The standard counting arguments give:

$a + b + c + d + e = 31$ ... (1)

$a + 2b + 3c + 4d + 5e = 60$ ... (2)

$2b + 6c + 12d + 20e = 90$ ... (3)

From these we can deduce

$c = -3b - 6a + 115;$

$d = 8a + 3b - 135;$

$e = -3a - b + 51;$

Since $d \geq 0$, we get $e \leq 2$.

Theorem(3.3.13): There are Rédei –type minimal blocking sets of size ten in PG(2, 5) .

Proof. Let B be a blocking set with $e>0$, $d\neq 0$. Let l be a 4-secant to B. and assume l is the line at infinity of the corresponding affine plane(z=0), and let{P1, P2}be the points l\ B. By Lemma (3.2.6), there is a point $O\notin B$ such that OP_1, OP_2 are bisecants to B . Let U_1,U_2 be the remaining points of B ,and assume $P_1 =(1,0,0)$, $P_2 =(0,1,0)$,$O=(0,0,1)$,$U_1=(1,1,1)$.Nowthe affine lines joining OP_1, OP_2 are y=0,x=0. The lines joining P_1U_1,P_2U_2 either tangent to B or pass through U_2.On OP_1 ; Y =0, we need to select two points of the set $l_{1=}\{(1,0,-2),(1,0,-1), (1,0,2)\}$, and on OP_2 ;X =0, we need to select two points of the set $l_{2=}\{(0,1,-2),(0,1,-1), (0,1,2)\}$.Choose(1,0,- 2),(1,0,- 1) from l_1, and (0,1,-2), (0,1,2), and $U_2=(1,-2,2)$ with the four points at z=0 in B and $U_{1\,,}$ these ten points form minimal blocking set.

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