J.Edu. Sci , Vol. (19) No.(3) 2007

Existence Of (18,9;f)-Arc Of Type (4,9) In PG(2,5)

Makbola J.
Civil Engg. Department/College of Engineering/University of Mosul

Received	Accepted
3/9/2006	21/2/2007

الخلاصة

تم في هذا البحث اثبات وجود القوس –(18,9;f) من النوع (4,9) عندما $L_0=13$ في المستوي الاسقاطي ذي الرتبة الخامسة وأعطينا وصف شامل له مع مثال تفصيلي .وباستخدام الحاسبة الشخصية تم بناء بعض اقواس –(13,4) في المستوي الاسقاطي PG(2,5) ومقارنة هذه الاقواس مع القوس –(18,9;f) من النوع (4,9) ، وتضمن البحث ايضا براهين النظريات التي تم استنتاجها .

Abstract

In this paper we prove the existence of (18,9;f)-arc of type (4,9) when $L_0=13$ in the projective plane of order five ,and classified it then give an example of this case .Then by personal computer we construct some projectively distinct (13,4)-arc in PG(2,5) and compare the results with (18,9;f)-arc of type (4,9) .Also this paper conclude the proves of the theorems that deduced.

1. Introduction:

Let PG(2,q) be a projective plane π of order q ,a (k,n)-arc A in the projective plane is a set of k points ,such that some n, but no n+1 of them are collinear.
The following lemma are well-known and the proof found in [8].

Lemma-1:

Let T_i denote the total number of i-secants of A in the plane π and R_i the number of i-secants of A through a point p in the plane, and S_i the number of i-secants to A through a point Q of $\pi \backslash A$,then for a (k,n)-arc A the following equations holds:

$$\sum_{i=0}^{n} T_i = q^2+q+1 \qquad \text{……………..(1-1)}$$

$$\sum_{i=1}^{n} iT_i = k(q+1) \qquad \text{……………..(1-2)}$$

$$\sum_{i=2}^{n} \frac{i(i-1)T_i}{2} = \frac{k(k-1)}{2} \qquad \text{……………..(1-3)}$$

$$\sum_{i=1}^{n} R_i = q+1 \qquad \text{……………..(1-4)}$$

$$\sum_{i=2}^{n} (i-1)R_i = k-1 \qquad \text{………………(1-5)}$$

$\sum_{i=0}^{n} S_i = q+1$ ………………(1-6)

$\sum_{i=1}^{n} iS_i = k$ ………………(1-7)

$iT_i = \sum_{p} R_i$ …..…………..(1-8)

$(q+1-i)T_i = \sum_{Q} S_i$ ……………..(1-9)

If f is a function from the set of points of the projective plane π into the set of natural number N ,the value f(p) is called the weight point p and if F is a function from the set of lines of π into N, the value F(r) is called the weight line r , that is $F(r) = \sum_{p \in r} f(p)$.

A (k,n;f)-arc K of π is a set of k points such that K does not contain any points of weight zero. The line r of π is called i-secant if the total weight of r is i . L_j denotes the number of points having weight j for j=0,1,2,…,w and we used V_i^j for the number of lines of weight i through a point of weight j ,we also denote the number of lines of weight i by t_i ,the integers t_i are called the characters of K .If the points in the plane are only of weight zero and one ,then K is a (k,n)-arc.

The development of the theory of (k,n;f)-arcs is due ,essentially, to D'Agostini [1] & [2]. Also [3] &[5] proves the existence of this arc for different projective planes. [6] study (k,n;f)-arcs of type (m,n) in PG(2,5) and show that this arc does not exist when L_0 =13.

In this paper we prove the existence of this arc when L_0 = 13, and classified it, then give an examples of this case.

Let W denote the total weight of K ,so by [1] we have :

$m(q+1) \leq W \leq (n-w)(q+1) + w$ …………………(1-10)

Arcs for which equality holds on the left are called minimal and arcs for which equality holds on the right are called maximal .Also [1] proved to be a necessary condition for the existence of a (k,n;f)-arc K of type (m,n) , 0 < m < n is that

$q \equiv 0 \mod(n-m)$ …………………..(1-11)

And

$w \leq n-m$ …………………..(1-12)

Theorem-1 :

Let K be a (k,n;f)-arc of type (t_1=m,…,t_r=n) for which the maximum weight of any point is w . Define a new (k',n';f')-arc K' by f'(p)= w–f(p) for each point p in the plane. Then K' is maximal iff , K is a minimal .

Proof : See [6]

Theorem-2 :

For a minimal (k,n;f)-arc of type (n–5,n) in PG(2,5) ,we have :

$$V_{n-5}^0 = q+1 \qquad V_n^0 = 0$$

$$V_{n-5}^1 = \frac{4}{5}q+1 \qquad V_n^1 = \frac{1}{5}q$$

$$V_{n-5}^{2}=\frac{3}{5}q+1 \qquad V_{n}^{2}=\frac{2}{5}q$$

$$V_{n-5}^{3}=\frac{2}{5}q+1 \qquad V_{n}^{3}=\frac{3}{5}q \qquad \text{.......(1-13)}$$

$$V_{n-5}^{4}=\frac{1}{5}q+1 \qquad V_{n}^{4}=\frac{4}{5}q$$

$$V_{n-5}^{5}=1 \qquad V_{n}^{5}=q$$

Proof : See[4]

2. (k,n;f)-arcs of type (n–5,n) with $L_i > 0$,i=0,1,2 & L_j=0, j=3,4,5:

Let t_{n-5} be the number of lines of weight n-5 and t_n be the number of lines of weight n, then

$t_{n-5} + t_n = q^2 + q + 1$ ……….(2-1)

$(n-5)\, t_{n-5} + n\, t_n = w(q+1) = (n-5)(q+1)^2$ ……….(2-2)

Solving (2-1) & (2-2) gives

$t_n = (1/5)(n-5)q$ ………(2-3)

$t_{n-5} = (1/5)(5q^2 + 10q - nq + 5)$ ……….(2-4)

Now let M be an n-secant which has no point of weight 0 and suppose that on M there are α points of weight 2 and β of weight 1, then counting points of M gives:

$$\alpha + \beta = q + 1$$

and the weight of points on M gives:

$$2\alpha + \beta = n$$

that is

$$\left.\begin{aligned}\alpha &= n-(q+1)\\ \beta &= 2(q+1)-n\end{aligned}\right\} \qquad \text{...........(2-5)}$$

Counting the incidences between points of weight 1 and n-secants gives :

$$L_1 V_n^1 = t_n \beta$$

By using (1-13) and the equations (2-3) and (2-5) we have

$$L_1 = (n-5)(2q + 2 - n) \qquad \text{...........(2-6)}$$

Similarly ,counting incidences between points of weight 2 and the n-secants gives:

$$L_2 V_n^2 = t_n \alpha$$

Hence, using (1-13) and the equations (2-3) and (2-5) we have

$$L_2 = [(n-5)(n-q-1)]/2 \qquad \text{...........(2-7)}$$

Since

$$L_0 + L_1 + L_2 = q^2 + q + 1$$

Then by equations (2-6) and (2-7) we get

$$2q^2 - (17-3n)q + n^2 - 8n + 17 - 2L_0 = 0 \qquad \text{............(2-8)}$$

3- The case when the number of points of weight 0 is thirteen :

Applying equation (2-8) when $L_0 = 13$, we get

$2q^2 - (17 - 3n)q + n^2 - 8n - 9 = 0$ …………(3-1)

we know that the discriminant of algebraic equation of degree two is $\Delta = b^2 - 4ac$,where $b = (17-3n)$,$a = 2$,and $c = n^2 - 8n - 9$,then Δ becomes $\Delta = (n - 19)^2$

From equations (2-6),(2-7) ,it is clear that $5 < n \le 12$, and all the possibilities of solutions of (3-1) are given in the following table-1 .

Table-1

n	6	7	8	9	10	11	12
Δ	169	144	121	100	81	64	49
q	1.5 , – 7	4 , – 2	– 1 , 4.5	0 , 5	1 , 5.5	2 , 6	3 , 6.5

Hence the only solution of these compatible with n & Δ being non-negative integers and $q \equiv 0 \bmod(n\text{-}m)$ is $\Delta = 100$ and $n = 9$.

From equation (1-10) and equation (1-12) ,we get the following for our case:

1) $1 \le w \le 5$
2) $24 \le W \le 44$

From theorem-1 we get the maximum case.

Now we discuss the minimum case when W=24 ,from theorem-2 we have the following results:

$$\left.\begin{array}{ll} V_4^0 = 6 & V_9^0 = 0 \\ V_4^1 = 5 & V_9^1 = 1 \\ V_4^2 = 4 & V_9^2 = 2 \end{array}\right\} \quad \text{……..(3-2)}$$

Since 9-secant does not contain any point of weight 0 ,then we give the following lemma:

Lemma-2:

No point of weight zero lies on any 9- secant of a (18,9;f)-arc of type (4,9) .

Proof: From (table-1) when n=9. It is clear by theorem-2 and equation (3-2) , $V_9^0 = 0$.

Theorem-3:

No five points of weight zero can be collinear.

Proof: Suppose that there is a 4-secant line r, such that r have five points of weight zero. Then the other point on r has at most weight 2 ,means that the weight of r is two which is a contradiction.

Corollary:

The points of weight 0 form a (13,4)-arc

Lemma-3:

For the existence of(18,9;f)-arc K of type (4,9) and the points of weight 0 form (13,4)-arc in PG(2,5), we must have the following:

1) The number t_9 of 9-secants of K is four
2) The number t_4 of 4-secants of K is twenty seven
3) The number L_2 of points of weight 2 is six
4) The number L_1 of points of weight 1 is twelve

Proof: Follows from equations (2-3),(2-4),(2-6) & (2-7).

4- Classification of the lines of the plane with respect to an (18,9;f)-arc of type (4,9) :

Let X be an 9-secant of (18,9;f)-arc K , since $V_9^{0} = 0$, then on X there are only points of weight 2 and points of weight 1 **,** suppose that on X there are α points of weight 2 and β points of weight 1, then

$\alpha + \beta = 6$

Counting the points of weights 2 & 1 on X ,we get

$2\alpha + \beta = 9$

Hence the unique solution of these equations is

$\alpha = 3$ **,** $\beta = 3$

Suppose that r_1 be a 4-secant having one point of weight 0**,** α points of weight 2 and β points of weight 1 **,** then

$\alpha + \beta = 5$
$2\alpha + \beta = 4$

There is no solution of these equations. Hence there does not exist 1-secant of K

Suppose that r_2 be a 4-secant having two points of weight 0 **,** α points of weight 2 and β points of weight 1 **,** then

$\alpha + \beta = 4$
$2\alpha + \beta = 4$

So $\alpha = 0$, $\beta = 4$

Suppose that r_3 be a 4-secant having three points of weight 0, α points of weight 2 and β points of weight 1 **,** then

$\alpha + \beta = 3$
$2\alpha + \beta = 4$

So $\alpha = 1$, $\beta = 2$

Suppose that r_4 be a 4-secant having four points of weight 0 **,** α points of weight 2 and β points of weight 1 **,** then

$\alpha + \beta = 2$
$2\alpha + \beta = 4$

So $\alpha = 2$, $\beta = 0$

These are the possible solutions

From the above we conclude the following table-2

Table-2

Type of the lines	Points of weight 0	Points of weight 1	Points of weight 2
r_2	2	4	0
r_3	3	2	1
r_4	4	0	2
X	0	3	3

Hence , we have proved the following lemma-4 :

Lemma -4:
The lines of PG(2,5) are partitioned into four classes with respect to a minimal (18,9;f)-arc of type (4,9) as follows:

1) r_2 which contains two points of weight 0 ,and four points of weight 1
2) r_3 which contains three points of weight 0 ,two points of weight 1 and one points of weight 2
3) r_4 which contains four points of weight 0 ,and two points of weight 2
4) X which contains three points of weight 1 ,and three points of weight 2

Corollary -1:
There is no point of weight 1 on the 4-secant of (13,4)-arc

Corollary -2:
There is no point of weight 2 on the 2-secant of (13,4)-arc

5- The Projectively distinct (13,4)-arc in PG(2,5):

Since the numbers of probabilities for finding the projectively distinct (k,n)-arcs is very large , it is impossible to construct these arcs by hand, so we used computer program . The program used in Fortran 77 which was taken from [7] and update it to be suitable for the plane PG(2,5) , then we find some projectively distinct (13,4)-arc listed in the following table-3

Table-3

(13,4)-arc Y_i	Points of Y_i													Number of T_4
	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}	
Y_1	29	30	0	22	1	3	6	8	10	14	17	23	27	5
Y_2	29	30	0	22	3	4	6	8	10	14	17	23	27	4

Y_3	29	30	0	22	2	3	4	5	6	13	14	23	28	8
Y_4	29	30	0	22	1	2	3	4	13	14	17	23	28	7
Y_5	29	30	0	22	2	3	4	5	6	14	17	23	28	8
Y_6	29	30	0	22	1	2	3	4	8	12	15	17	18	6
Y_7	29	30	0	22	1	2	3	6	9	12	23	24	26	3
Y_8	29	30	0	22	1	2	3	4	6	9	14	17	23	6
Y_9	29	30	0	22	1	2	3	4	6	10	14	17	23	7
Y_{10}	29	30	0	22	1	2	3	4	6	14	17	23	28	9
Y_{11}	29	30	0	22	2	6	13	14	18	21	23	27	28	5

6- Notations on (13,4)-arc:

Let $p \in K$ and suppose that through p there are α (4-secant) , β (3-secant), γ (2-secant) and δ (1-secant) then by using equations (1-4) & (1-5) of lemma-1 we get

$\alpha + \beta + \gamma + \delta = 6$ ……….(6-1)

$3\alpha + 2\beta + \gamma = 12$ ……….(6-2)

From (6-1) & (6-2) we have seven non-negative solutions explained in table-4 below :

Table-4

Type of the points	α	β	γ	δ
type - 1	0	6	0	0
type - 2	1	4	1	0
type - 3	2	2	2	0
type - 4	2	3	0	1
type - 5	3	0	3	0
type - 6	3	1	1	1
type - 7	4	0	0	2

If there are:

X_1 points of type 1

X_2 points of type 2

X_3 points of type 3

X_4 points of type 4

X_5 points of type 5

X_6 points of type 6

X_7 points of type 7

Then from equation (1-8) of lemma-1 and (table-4) , we have:

$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 = 13$ ………...(6-3)

$X_2 + 2X_3 + 2X_4 + 3X_5 + 3X_6 + 4X_7 = 12$ ………...(6-4)

$6X_1 + 4X_2 + 2X_3 + 3X_4 + X_6 = 54$ …………(6-5)

$X_2 + 2X_3 + 3X_5 + X_6 = 12$ ……….. (6-6)

$X_4 + X_6 + 2X_7 = 0$ …………(6-7)

Let $z \notin K$ and suppose that through z there are

α (4-secant), β (3-secant), γ (2-secant), δ (1-secant) and ε (0-secant) then by using equations (1-6) & (1-7) of lemma-1 we have

$\alpha + \beta + \gamma + \delta + \varepsilon = 6$ ………… .(6-8)

$4\alpha + 3\beta + 2\gamma + \delta = 13$ ………… .(6-9)

Equations (6-8) & (6-9) have sixteen non-negative integral solutions explained in table-5 below :

Table-5

Type of the point	4-secant α	3-secant β	2-secant γ	1-secant δ	0-secant ε
D_1	0	4	0	1	1
D_2	0	3	2	0	1
D_3	0	3	1	2	0
D_4	0	2	3	1	0
D_5	0	1	5	0	0
D_6	1	3	0	0	2
D_7	1	2	1	1	1
D_8	1	2	0	3	0
D_9	1	1	3	0	1
D_{10}	1	1	2	2	0
D_{11}	1	0	4	1	0
D_{12}	2	0	2	1	1
D_{13}	2	1	1	0	2
D_{14}	2	1	0	2	1
D_{15}	2	0	1	3	0
D_{16}	3	0	0	1	2

Suppose there are β_i points of type D_i , i = 1, ..., 16 then counting the point of the
plane $\pi \backslash A$, when A is a (13,4)-arc in PG(2,5) obtain the following :

$$\sum_{i=1}^{16} \beta_i = |\pi \setminus A| = 18 \qquad …….(6\text{-}10)$$

By using equation (1-9) of lemma-1 and (table-5) the following equations are obtained:

$\beta_6+\beta_7+\beta_8+\beta_9+\beta_{10}+\beta_{11}+2\beta_{12}+2\beta_{13}+2\beta_{14}+2\beta_{15}+3\beta_{16}$
$= 2T_4$ …….(6-11)

$4\beta_1+3\beta_2+3\beta_3+2\beta_4+\beta_5+3\beta_6+2\beta_7+2\beta_8+\beta_9+\beta_{10}+\beta_{13}+\beta_{14}$
$= 3T_3$ ……(6-12)

$2\beta_2+\beta_3+3\beta_4+5\beta_5+\beta_7+3\beta_9+2\beta_{10}+4\beta_{11}+2\beta_{12}+\beta_{13}+\beta_{15}$
$= 4T_2$ …….(6-13)

$\beta_1+2\beta_3+\beta_4+\beta_7+3\beta_8+2\beta_{10}+\beta_{11}+\beta_{12}+2\beta_{14}+3\beta_{15}+\beta_{16}$

$= 5T_1$ ……. (6-14)

$\beta_1 + \beta_2 + 2\beta_6 + \beta_7 + \beta_9 + \beta_{12} + 2\beta_{13} + \beta_{14} + 2\beta_{16} = 6T_0$ ...(6-15)

From equations (1-1), (1-2) & (1-3) of lemma-1 ,we get

$T_0 + T_1 + T_2 + T_3 + T_4 = 31$ …………..(a)
$T_1 + 2T_2 + 3T_3 + 4T_4 = 78$ …………..(b)
$T_2 + 3T_3 + 6T_4 = 78$ …………..(c)

From (table-3) we have $T_4 \leq 9$,then the solutions of the equations (a) ,(b) & (c) are listed in the following table-6

Table - 6

T_4	T_3	T_2	T_1	T_0
5	14	6	4	2
4	17	3	5	2
8	6	12	4	1
7	8	12	2	2
8	7	9	7	0
6	13	3	9	0
3	18	6	0	4
6	11	9	3	2
7	8	12	2	2
9	4	12	6	0
5	13	9	1	3

Lemma-5:

The points of weight 2 of the (18,9;f)-arc K of type (4,9) are points of type D_6, D_{13} & D_{16} when the point of weight 0 form a (13,4)-arc.

Proof: From equation (3-2) through a point of weight 2 there pass two 9-secants of K, suppose Q is a point of type D_8 ,and suppose having weight 2, from (table-5) the following pass through Q one 4-secants ,two 3-secants and three 1-secants.

But lemma-4 shows that the i-secants (i=1,2,3,4) of a (13,4)-arc are 4-secants of K and the 0-secants of a (13,4)-arc which have point of weight 2 are 9-secants of K.

Hence through Q there pass six 4-secants of K and zero 9-secants of K . Which is a contradiction because $V_4^2 = 4$, $V_9^2 = 2$
That is the points of type D_8 are not points of weight 2.

By the same way we can show that the points of type D_i , i = 1,...,15, i $\neq$ 6 , 13. Suppose p is a point of type D_6 and suppose it has weight 2 ,from (table-5) through p there pass one 4-secants three 3-secants and two 0-secants of (13,4)-arc ,we showed in lemma-4 that the i-secants (i = 1 , 2 , 3 ,4) (which are 4-secants of K) and the two 0-secants of the (13,4)-arc which are either 9-secants or 4-secants with respect to K according to p is a point of weight 2 or weight 1 respectively .Hence p is possibly a point of weight 2. Similarly we can prove for D_{13} & D_{16}.

Corollary:

Let K be a (18,9;f)-arc of type (4,9) , then the points of weight 1 of K are points of type D_i of K , i=1,...,16 .

Example: An example may be found in PG(2,5) of (18,9;f)-arc of type (4,9) when the points of weight 0 form (13,4)-arc shown in (table-7)

Remarks on (table-7):

1) The points are marked inside ellipse are the points of weight 0
2) The underlined points are points of weight 2
3) The other points are points of weight 1.

Table-7

Lines	Points						Type	The equation	$W(L_j)$
L_1	0	3	**5**	12	**20**	30	4-secant	x = 0	4
L_2	1	**4**	6	13	21	0	3-secant	x + y = 0	4
L_3	2	**5**	7	14	22	1	3-secant	x + 2y +2z = 0	4
L_4	3	6	**8**	**15**	23	2	4-secant	x+2y – z=0	4
L_5	**4**	7	9	16	24	3	3-secant	x + y + 2z = 0	4
L_6	**5**	**8**	10	17	25	**4**	0-secant	x + y + z = 0	9
L_7	6	9	11	18	26	**5**	3-secant	x – 2y –2z = 0	4
L_8	7	10	12	19	27	6	2-secant	x – y + 2z = 0	4
L_9	**8**	11	13	**20**	**28**	7	0-secant	x – 2y +2z = 0	9
L_{10}	9	12	14	21	29	**8**	3-secant	y – 2z = 0	4
L_{11}	10	13	**15**	22	30	9	3-secant	x – z = 0	4
L_{12}	11	14	16	23	0	10	2-secant	x – 2y = 0	4
L_{13}	12	**15**	17	24	1	11	3-secant	x – 2y – z = 0	4
L_{14}	13	16	18	25	2	12	2-secant	x + 2y + z = 0	4
L_{15}	14	17	19	26	3	13	2-secant	x – y – 2z = 0	4
L_{16}	**15**	18	**20**	27	**4**	14	0-secant	x + y – z = 0	9
L_{17}	16	19	21	**28**	5	**15**	0-secant	x – y –z = 0	9

L_{18}	17	**20**	(22)	(29)	(6)	16	3-secant	y – z = 0	4
L_{19}	18	21	(23)	(30)	7	17	2-secant	x + 2z = 0	4
L_{20}	19	(22)	(24)	(0)	**8**	18	3-secant	x – y = 0	4
L_{21}	**20**	(23)	25	(1)	(9)	19	3-secant	x – y + z = 0	4
L_{22}	21	(24)	(26)	(2)	10	**20**	3-secant	x + 2y – 2z = 0	4
L_{23}	(22)	25	27	(3)	11	21	2-secant	x – 2y + z = 0	4
L_{24}	(23)	(26)	**28**	**4**	(12)	(22)	4-secant	x + y–2z = 0	4
L_{25}	(24)	27	(29)	5	13	(23)	3-secant	y + z = 0	4
L_{26}	25	**28**	(30)	(6)	14	(24)	3-secant	x + z = 0	4
L_{27}	(26)	(29)	(0)	7	**15**	25	3-secant	y = 0	4
L_{28}	27	(30)	(1)	**8**	16	(26)	3-secant	x –2z = 0	4
L_{29}	**28**	(0)	(2)	(9)	17	27	3-secant	x +2y = 0	4
L_{30}	(29)	(1)	(3)	10	18	**28**	3-secant	y +2z = 0	4
L_{31}	(30)	(2)	**4**	11	19	(29)	3-secant	z = 0	4

7- Existence of the (18,9;f)-arc of type (4,9) :

From the example in (table-7) we see that all the points of weight 2 are of type D_6 and all the points of weight 1 are of type D_2 , that is $D_i = 0$ for $i = 1,\dots,16$, $i \neq 2,6$.
Then the only solution of equations (6-10) , … , (6-15) is $\beta_2=12$, $\beta_6=6$, $\beta_i=0$ for $i=1,\dots,16$, $i \neq 2 , 6$

8- The case of (13,4)-arc with three 4-secants :

When X ={29 , 30 , 0 , 22 , 1 , 2 , 3 , 6 , 9 , 12 , 23 , 24 , 26} and by using (table-6) , we get: $T_4 =3$, $T_3 =18$, $T_2 = 6$,$T_1 =0$, $T_0 = 4$ and the solutions of (6-3) , … , (6-7) are :
$X_1 = 4$, $X_2 = 6$, $X_3 = 3$, and $X_4 = X_5 = X_6 = X_7 = 0$.
That is : there are four points of (13,4)-arc of type 1 , these points are :
1 = (1,– 1,– 2) , 9 =(1 ,2 , 1) , 24 = (1 ,1,– 1) & 29 = (1, 0 ,0) ,six points of type 2 which are :
0 =(0 ,0 ,1) , 2 =(1, 2, 0) , 6 =(1,– 1,– 1) , 22=(1, 1, 1) , 26 =(1,0,– 2) & 30 =(0, 1 ,0) , and the three points of type 3 are : 3 =(0 ,1 ,2) , 12 =(0, 1,– 2) & 23 =(1,– 2, 2) ,
see (table-7) .

Again from (table-7) the twelve points of weight 1 which are of type D_2 are :
7 = (1, 0, 2) , 10 = (1,– 2, 1) , 11= (1,– 2, 0) , 13 = (1,– 1, 1) , 14 = (1,– 2,– 1) ,
16 = (1,– 2,– 2) , 17 = (1, 2, 2) , 18 = (1, 1, 2) , 19 = (1, 1, 0) , 21 = (1,– 1, 2) ,
25=(1, 0,– 1) , 27= (1, 2,– 2) , and the six points of weight 2 which are of type D_6 are :

4 =(1,– 1, 0) , 5 =(0, 1 – 1) , 8 =(1, 1,– 2) , 15 =(1, 0, 1) , 20 =(0, 1, 1) ,
28 =(1, 2,– 1) .

REFERENCES

1. D'Agostini , E." Sulla caratterizzazione delle (k,n;f)-calotte di tipo (n-2,n) " Atti Sem.Mat.Fis.Univ. Modeno XXIX,(1980),263-275
2. D'Agostini , E. "On caps with weighted points in PG(2,q)" Discrete Mathematics 34, 103 -110 (1981).
3. Makbola ,J. and Raida ,D. "(k,n;f)-arcs in Galois plane of order seven" Basrah J.Science,Vol.13,N0.l ,49-56, 1995
4. Wilson , B.J. "(k,n;f)-arc and caps in finite projective spaces" Annals of Discrete mathematics 30 , 355-362(1986)
5. Hameed,F.K. " Weighted (k,n)-arcs in the projective plane of order nine" PH.D. Thesis,University of London(1989).
6. Mahmood ,R.D. "(k,n;f)-arcs of type (n-5,n) in PG(2,5)" M.SC. Thesis ,College of Science,University of Mosul(1990).
7. Nada,Y.K. " Lower bounds of t-fold blocking sets and construction of (k,n)-arcs in the Desargusian plane PG(2,q) " PH.D. Thesis,University of Mosul,(2006)
8. Hirschfeld ,J.W.P."Projective Geometries Over Finite Fields" ,Oxford,(1979).