On *P*-Injective Modules And γ-Regular Rings

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الملخص

في هذا البحث درسنا الحلقات المنتظمة من النمط -γ والعلاقـة بـين هـذه الحلقـات والحلقات المنتظمة الضعيفة، والحلقات المنتظمة من النمط -π ، والحلقات المنتظمة القوية من النمط -π والحلقات الشبه منتظمة من النمط -π وحلقات-SNI. كمـا درسـنا الموديـولات الغامرة من النمط -P ووجدنا العلاقة بين الموديولات الغـامرة مـن الـنمط -P والحلقـات المنتظمة من النمط -γ.

ABSTRACT

The main purpose of this paper is to study γ-regular rings, and the connection between such rings and weakly regular, π-regular, strongly π-regular, semi π-regular and SNI-rings. We also study *P*-injective modules, and to find it's relation with γ-regular rings.

1. Introduction

Throughout this paper, *R* denotes an associative ring with identity and all modules are unitary. An ideal *I* of a ring *R* is called reduced if it contains no non-zero nilpotent elements. For any non empty subset *x* of a ring *R*, the right (left) annihilator of *x* will be denoted by *r(x)* *(l(x))*, respectively. Recall that:

(1) A ring *R* is said to be Von Neumann regular if for every $a \in R$, there exists $b \in R$ such that $a=aba$. The concept of regular rings was introduced by J. Von Neumann in 1936.

(2) A ring *R* is said to be right (left) weakly regular if $a \in aRaR$ $(a \in RaRa)$ for every $a \in R$, *R* is weakly regular ring if it is both right and left weakly regular.

(3) A ring *R* is said to be π-regular if for every element *a* in *R* there exists a positive integer $n=n(a)$ depending on *a*, such that $a^n \in a^nRa^n$. [8]

(4) A ring R is said to be right semi π-regular if for each a in R, there exist a positive integer n and an element b in R such that $a^n=a^nb$ and $r(a^n)=r(b)$. [7]
(5) A ring R is called a right (left) self-injective (and is denoted by SNI-rings), if and only if, for any essential right (left) ideal E of R, every right (left) R-homomorphism of E in to R extends to one of R in to R. [3]
(6) A ring R is said to be strongly regular if for every $a \in R$, there exists $b \in R$ such that $a=a^2b$.
(7) A ring R is said to be strongly π-regular if for every $a \in R$, there exists $b \in R$ and a positive integer n such that $a^n=a^{n+1}b$.
(8) Let I be an ideal of a ring R. We say that I is pure if for all $x \in I$ there exists $y \in I$ such that $x=xy$ A ring R is called a right (left) semi-duo, if and only if, every principal right (left) ideal of R is a two sided ideal generated by the same element.
(9) A right R-module M is said to be P-injective, if and only if, for each principal right ideal I of R and every right R-homomorphism $f:I\rightarrow M$, there exists y in M such that $f(x)=yx$ for all $x \in I$.
(10) A right R-module is called GP-injective if for any $0\neq a \in R$, there exists a positive integer n such that $a^n\pm 0$, and any right R-homomorphism of a^nR in to M extends to one of R in to M.

2. γ-Regular Rings

In this section we introduce the definition of γ-regular rings and we discuss the connection between γ-regular rings and other rings which reduced.

Definition 2.1: [5]

An element a of a ring R is said to be γ-regular if there exists b in R and a positive integer $n\neq 1$ such that $a=ab^na$. A ring R is said to be γ-regular if every element of R is γ-regular element.

Examples 2.2:

The following rings are γ-regular rings:

1- Z_3, Z_5, Z_{11}, Z_{15}
2- Let $R(Z_2)$ be the ring of all 2 by 2 matrices over the ring Z_2 (the ring of integer module 2) which are strictly upper triangular. Clearly, $R(Z_2)$ is γ-regular ring.

We define a condition (*) as follows:

Definition 2.3: [5]

A ring R satisfies condition (*) if for every $1\neq a\in R$ and $b \in R$, there exists a positive integer $m > 1$ such that $ab=b^ma$.

Theorem 2.4:

Let R be a reduced ring satisfies condition (*) then the following are equivalent:

1- R is a γ-regular ring.
2- R is a strongly π-regular ring.
3- R is π-regular ring.

Proof:

1⇒ 2: Since R is γ-regular ring and satisfies condition (*), then by [5, Theorem 4.4] R is strongly regular ring. So for every $a \in R$ there exists $b \in R$ such that $a=a^2b$. So $(1-ab) \in r(a)$. By [1; Lemma(2.1.9)] $r(a)=r(a^n)$, whence $a^n(1-ab)=0$, then $a^n=a^{n+1}b$. Therefore R is strongly π-regular ring.

2⇒ 1: For every $a \in R$ there exists $n \in Z^+$ and element $b \in R$ such that $a^n=a^{n+1}b$. Now since R satisfies condition (*), then for every $a,b \in R$, $ab=b^m a$ for some positive integer $m > 1$. Then $a^n=a^n b^m a$. So $(1-b^m a) \in r(a^n)$. By [1; Lemma(2.1.9)], $r(a^n)=r(a)$, whence $a(1-b^m a)=0$, then $a=ab^m a$. Therefore R is γ-regular ring.

1⇒ 3: Since R is γ-regular ring and satisfies condition (*), then R is strongly π-regular ring, and since R is reduced, then by [1; corollary(2.2.7)] R is π-regular ring.

3⇒ 1: Trivial. ■

Theorem 2.5:

Let R be a strongly π-regular ring satisfies condition (*). Then any reduced ideal of R is γ-regular.

Proof:

Let I be any reduced ideal of R, and let $a \in I$. Since R is strongly π-regular, there exists a positive integer n and an element b in R such that $a^n=a^{n+1}b$ which implies $a^n(1-ab)=0$ and hence $(1-ab) \in r(a^n)=r(a)=l(a)$, gives $(1-ab)a=0$. Therefore $a=aba$. Now let $c=bab \in I$, then $aca=a.bab.a=aba=a$. Thus $a=aca$, $c \in I$. Consider $(a-a^2c)^2=a^2-a^3c-a^2ca+a^2ca^2c$

$=a^2-a^3c-a(aca)+a(aca)ac$

$=a^2-a^3c-a^2+a^3c$

$=0$ But I is reduced, then $a-a^2c=0$ implies that $a=a^2c$. Thus I is a strongly regular ideal. Since R satisfies condition (*), then by [5; Theorem 4.4] I is γ-regular ideal. ■

Theorem 2.6:

If R is π-regular ring satisfies condition (*) and all idempotent elements of R are central, then R is γ-regular ring.

Proof:

Since R is π-regular ring and all idempotent elements of R are central, then by [*1* ; Corollary (2.2.10)] R is strongly π-regular, and since R satisfies condition (*) then by [Theorem 2.4] R is γ-regular ring. ■

Theorem 2.7:

Let R be a left semi-duo ring satisfies condition (*). Then R is γ-regular if for every $a \in R$, there exists $n \in Z^+$ such that a^nR is a right semi-regular ideal.

Proof:

Since R is a left semi-duo ring and for every $a \in R$, there exists $n \in Z^+$ such that a^nR is a right semi-regular ideal, then by [1; Theorem (2.3.8)] R is π-regular ring, and since R satisfies condition (*) then by [Theorem 2.4] R is γ-regular ring. ■

Theorem 2.8:

Let R be a duo ring satisfies condition (*). Then R is γ-regular if for all $a \in R$, there exists a positive integer n such that a^nR is a pure ideal.

Proof:

Since R be a duo ring and for all $a \in R$, there exists a positive integer n such that a^nR is a pure ideal, then by [1; Theorem (2.3.5)] R is π-regular ring, and since R satisfies condition (*) then by [Theorem 2.4] R is γ-regular ring. ■

Theorem 2.9:

Let R be a reduced ring satisfies condition (*). Then R is γ-regular iff for all $a \in R$ there exists unit element $k \in R$ and some idempotent $e \in R$ such that $a=ke$.

Proof:

Assume that R is γ-regular. For any $a \in R$ then there exists $b \in R$ and a positive integer $n \neq 1$ such that $a=ab^na$. Hence $e=ab^n$. Now, we shall prove that e is idempotent element, so $e^2=(ab^n)^2=ab^nab^n=ab^n=e;$ and since R is a reduced then e is a central so $a=ab^na=ea=ae$. If we set, $a=e-e+a=e-e^2+ae=(1-e+a)e=ke$. Where $k=(1-e+a)$, then $(1-e+a)(1-e+eb^n)=1-e+eb^n-e+e^2-e^2b^n+a-ae+aeb^n=1$. So $k=(1-e+a)$ is a unit element in R. Therefore $ke=(1-e+a)e=(1-e+a)ab^n=ab^n-eab^n+aab^n=ab^n-ab^n+ae=a$

Conversely, let $a \in R$, and $a=ke$ for some unit $k \in R$ and some idempotent $e \in R$. Hence $e=ax$ where x the inverse of k. Now $ae=aax=keax=kee=ke^2=ke=a$. Therefore $a=ae=aax$. Since R satisfies condition (*), then for every $a,x \in R$, $ax=x^ma$ for some positive integer $m > 1$. Then $a=ax^ma$. Therefore R is γ-regular ring. ■

Theorem 2.10:

Let R be a ring satisfies condition (*), if R is γ-regular ring. Then R is semi π-regular ring.

Proof:

Let R be a γ-regular ring satisfies condition (*). Then by [Theorem 2.4] R is π-regular ring, then by [1; Theorem (1.3.1)] R is a right and left semi-π regular ring. ■

Example 2.11:

Let $R(Z_2)$ be the ring of all 2 by 2 matrices over the ring Z_2 (the ring of integer modulo 2) which are strictly upper triangular. The elements of $R(Z_2)$ are:

$I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $0=\begin{bmatrix}0&0\\0&0\end{bmatrix}$, $A=\begin{bmatrix}1&1\\0&0\end{bmatrix}$, $B=\begin{bmatrix}1&1\\0&1\end{bmatrix}$, $C=\begin{bmatrix}0&1\\0&1\end{bmatrix}$, $D=\begin{bmatrix}0&1\\0&0\end{bmatrix}$, $E=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, *and* $F=\begin{bmatrix}0&0\\0&1\end{bmatrix}$

$R(Z_2)$ is γ-regular ring satisfies condition (*) clearly, $R(Z_2)$ is a right semi π-regular ring, however $D=DC=DF$ and $r(D)=r(C)=r(F)=\{0,A,D,E\}$.

Theorem 2.12:

Let R be a reduced semi π-regular ring satisfies condition (*) with every non-zero divisor has inverse. Then R is γ-regular ring.

Proof:

Since R is semi π-regular ring, then $r(a^n)=r(e)$ where e is central idempotent element. Let $x \in a^nR \cap eR$ implies that $x=a^nr$, and $x=er'$ for some $r,r' \in R$. Now, see that $x=er'=e.er'=ex$. Since $e \in eR=r(a^n)$ then $a^ne=ea^n=0$. Also $ex=ea^nr=0$, $x=a^nr$, then $x=ex=0$. Thus $a^nR \cap eR=0$. Now we must prove that (a^n+e) is non-zero divisor. Let $(a^n+e)y=0$ implies that $a^ny=-ey$. That is $a^ny=-ey \in a^nR \cap eR$. Since $a^nR \cap eR=0$. Then $a^ny=ey=0$ and we have $a^ny=0$. That is $y \in r(a^n)=eR$. There exists $r_1 \in R$ such that $y=er_1$, also $0=ey=e.er_1=e^2r_1=er_1=y$ (e is idempotent), since (a^n+e) is a non-zero divisor. Let x be the inverse of (a^n+e). Then we have $(a^n+e)x=1$ implies that $a^n(a^n+e)x=a^n$ implies $(a^{2n}+a^ne)x=a^n$. Since $a^ne=0$, then $a^{2n}x=a^n$ implies $a^n=a^na^nx=a^nxa^n$. Since R satisfies condition (*) then by [Theorem 2.4] R is γ-regular ring. ■

Theorem 2.13:

Let R be a commutative ring satisfies condition (*), then the following are equivalent:

1- R is weakly regular ring,

2- R is γ-regular ring.

Proof:

1⇒ 2: Since R is a commutative weakly regular ring, then by [2; Theorem (1.3.9)] R is regular ring, and since R satisfies condition (*), then by [5; Theorem 4.6] R is γ-regular ring.

2⇒ 1: Since *R* is γ-regular ring satisfies condition (*), then by [5; Theorem 4.6] *R* is regular ring, and since *R* is commutative then by [2; Theorem (1.3.9)] *R* is weakly regular ring. ■

Theorem 2.14:

If *R* is a right SNI-ring satisfies condition (*), then *R* is γ-regular ring.

Proof:

Since *R* is a right SNI-ring, then by [3; corollary 3.4; P 149] *R* is weakly regular ring, and since *R* satisfies condition (*), then by [Theorem 2.13] *R* is γ-regular ring. ■

Theorem 2.15:

If *R* is a right SNI-ring which is left self-injective, satisfies condition (*). Then *R* is γ-regular ring.

Proof:

Let *R* be a right SNI-ring. Then *R* is semi-prime ring. Thus for any left ideal I, $L(I) \cap I=0$. Since *R* is SNI-ring, then *R* is reduced ring, and for any non-zero element *a* in *R*, $r(a)=L(a)$. Thus $L(r(a)) \cap L(a)=L(L(a)) \cap L(a)=0$, and since *R* is left self-injective ring, then aR is a right annihilator and

$R =r(L(r(a)))+r(L(a))$

$=r(a)+aR$ In particular $1=d+ab$, for some *b* in *R*, and *d* in $r(a)$. Hence $a=aba$. Since *R* satisfies condition (*), then by [5; Theorem 4.6] *R* is γ-regular ring. ■

Theorem 2.16:

Let *R* be a ring satisfies condition (*), then *R* is γ-regular if and only if *R* is a right semi-regular ring.

Proof:

Since *R* is a right semi-regular ring, then by [6 ; Theorem 3.2] $r(a)$ is direct summand for every *a* in *R*, and since *R* satisfies condition (*), then by [5 ; Proposition 4.11] *R* is γ-regular ring.

Conversely: If *R* is γ-regular, then $r(a)$ is direct summand for every *a* in *R* [3; Proposition 2.17]. Therefore *R* is a right semi-regular ring [6; Theorem 3.2]. ■

3. *P*-Injectivity:

In this section we discuss the connection between *P*-injective with γ-regular ring.

Theorem 3.1:

Let R be a ring satisfies condition (*) with every principal right ideal a^nR is a right annihilator generated by the same element, and if R/a^nR is P-injective. Then R is γ-regular ring.

Proof:

Let a be a non-zero element in R. Define a right R-homomorphism $f:a^nR\rightarrow R/a^nR$ by $f(a^nx)=x+a^nR$; Clearly f is well defined. Since R/a^nR is P-injective, then there exists $c\in R$ such that $f(a^nx)=(c+a^nR)a^nx=ca^nx+a^nR$, for each element x in R. In particular; $f(a^n)=1+a^nR=ca^n+a^nR$, which implies $1-ca^n \in a^nR=r(a^n)$. Thus $1-ca^n \in r(a^n)$. Therefore, $a^n=a^nca^n$. Since R satisfies condition (*) then by [Theorem 2.4] R is γ-regular rin ■

Theorem 3.2:

Let a be an element of a left semi-duo ring, satisfies condition (*), and if R/a^nR is P-injective and a^nR is projective. Then R is γ-regular ring.

Proof:

Let $a \in R$. Define a right R-homomorphism $f:R/a^nR\rightarrow a^nR/a^{n+1}R$ by $f(y+a^nR)=a^ny+a^{n+1}R$ for all $y \in R$. Since a^nR is projective, there exists R-homomorphism $g:a^nR\rightarrow R/a^nR$ such that $f(g(a^nx))=a^nx+a^{n+1}R$ for all $x\in R$. But R/a^nR is P-injective, then there exists $c\in R$ such that $g(a^nx)=(c+a^nR)a^nx$ then $a^nx+a^{n+1}R=f(g(a^nx))$

$=f((c+a^nR)a^nx)$

$=f(ca^nx+a^nR)$, (since $a^nx \in a^nR$)

$=a^nca^nx+a^{n+1}R$ Since R is a left semi-duo ring, then $ca^n \in Ra^n=a^nR$, then $ca^n=a^nr$ for some $r \in R$. So $a^nx+a^{n+1}R=a^{2n}rx+a^{n+1}R$ implies $a^nR=a^{2n}R$. Thus $a^n=a^{2n}d$ for some $d \in R$. Therefore R is strongly π-regular ring. Since R satisfies condition (*) then by [Theorem 2.4] R is γ-regular rin ■

Theorem 3.3:

Let R be a reduced ring satisfies condition (*). Then R is γ-regular ring if $aR/(aR)^2$ is P-injective for all $a \in R$ such that $r(a) \subseteq (aR)^2$.

Proof:

Let $aR/(aR)^2$ is P-injective ring. Defined $f:aR\rightarrow aR/(aR)^2$ as a right R-homomorphism by $f(ax)=ax+(aR)^2$, for all x in R, then f is a well define right R-homomorphism. Indeed let $x_1,x_2 \in R$ with $ax_1=ax_2$, implies $(x_1-x_2) \in r(a) \subseteq (aR)^2$, thus $ax_1+(aR)^2=ax_2+(aR)^2$. Hence $f(ax_1)=ax_1+(aR)^2=ax_2+(aR)^2=f(ax_2)$. Since $aR/(aR)^2$ is P-injective, then there exists c in R such that $f(ax)=(ac+(aR)^2ax=acax+(aR)^2$, for all $x \in R$ yields $a+(aR)^2=f(a)=aca+(aR)^2$, so $(a-aca) \in (aR)^2$. Since $aca \in (aR)^2$, then $a \in (aR)^2$. Thus $a \in aRaR$. Let $ab_1,ab_2 \in aR$ for any two element $b_1,b_2 \in R$, then $a=ab_1ab_2=a(b_1ab_2)=a(b_1b_2)a=aca$ for some c in R. Thus $a=aca$. Since R satisfies condition (*), then by [5 ; Theorem 4.6] R is γ-regular ring. ■

Theorem 3.4:

Let R be a reduced ring satisfies condition (*) with every maximal right ideal is GP-injective. Then R is γ-regular ring.

Proof:

Let $a \in R$. We claim first $a^nR+r(a^n)=R$. If not, there exists a maximal right ideal M containing $a^nR+r(a^n)$. Define the canonical injective $f:a^nR\rightarrow M$ by $f(a^nb)=a^nb$ for any $b \in R$. Since M is GP-injective, then there exists $c \in R$ such that $f(a^nb)=ca^nb$. Therefore $a^n=f(a^n)=ca^n$. Thus $1-c \in L(a^n)=r(a^n) \subseteq M$, which implies $1 \in M$, a contradiction. Hence $a^nR+r(a^n)=R$. In particular $a^nc+d=1$ for some $c \in R$ and $d \in r(a^n)$. so $a^nca^n=a^n$. Since R satisfies condition (*), then by [Theorem 2.4] R is γ-regular ring. ■

Theorem 3.5:

Let R be a duo ring satisfies condition (*). Then R is γ-regular if for all $a \in R$ there exists a positive integer n such that the principal ideal a^nR is idempotent.

Proof:

Let I be an ideal of R such that $I=a^nR$ with $a \in R$ and $n \in Z^+$, assume that $I^2=I$, and let $a^n \in a^nR=(a^nR)^2$. Since R is a duo ring, then $a^n \in a^{2n}R$. Hence $a^n=a^{2n}c$ for some $c \in R$. Thus $a^n=a^n.a^nc=a^n.da^n$ for some $d \in R$. Therefore, R is π-regular ring. Since R satisfies condition (*), then by [Theorem 2.4] R is γ-regular ring. ■

Theorem 3.6:

Let R be a reduced ring satisfies condition (*). Then R is γ-regular if every principal right ideal is a right annihilator generated by an element in R and R/aR is P-injective ring.

Proof:

Let $0\neq a \in R$. Now, define a right R-homomorphism $f:aR\rightarrow R/aR$ by $f(ax)=x+aR$ for all $x \in R$, then f is well-define, indeed, let $ax_1=ax_2$ for any two elements x_1,x_2 in R, then $a(x_1\text{-}x_2)=0$. So $(x_1\text{-}x_2) \in r(a)=aR$, then $x_1+aR=x_2+aR$, it mean $f(ax_1)=x_1+aR=x_2+aR=f(ax_2)$. Now, since R/aR is P-injective then there exists $c \in R$ such that $f(ax)=(c+aR)ax$ for all $x \in R$. Now, $f(a)=1+aR=ca+aR$, implies $1\text{-}ca \in aR=r(a)$. So $1\text{-}ca \in r(a)$, whence $a(1\text{-}ca)=0$, then $a\text{-}aca=0$, so $a=aca$. Since R satisfies condition (*), then by [5; Theorem 4.6] R is γ-regular ring. ■

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