On Rings whose Maximal Essential Ideals are Pure

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ABSTRACT

This paper introduces the notion of a right MEP-ring (a ring in which every maximal essential right ideal is left pure) with some of their basic properties; we also give necessary and sufficient conditions for MEP – rings to be strongly regular rings and weakly regular rings.

Keywords: MEP-rings ,pure ideals ,weakly regular ring.

حول الحلقات التي فيها كل مثالي اعظمي اساسي يكون نقي

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الملخص

يقدم هذا البحث مفهوم الحلقات من نوع MEP (الحلقات التي فيها كل جزء مثالي ايمن أعظمي أساسي هو نقي أيسر) وإعطاء الخواص الأساسية لها. كذلك إعطاء الشروط الضرورية والكافية للحلقة MEP لكي تكون حلقة منتظمة بقوة ومنتظمة بضعف.

الكلمات المفتاحية: الحلقات من نوع MEP، مثالي نقي، حلقة منتظمة بضعف.

1- <u>Introduction</u>

An ideal I of a ring R is said to be right (left)pure if for every $a \in I$, there exists $b \in I$ such that a=ab (a=ba),[1],[2].

Throughout this paper, R is an associative ring with unity.

Recall that:

1) R is called reduced if R has no non _zero nilpotent elements.
2) For any element a in R we define the right annihilator of a by r(a)={ $x \in R : ax = 0$ } , and likewise the left annihilator l(a).
3) R is strongly regular [4], if for every $a \in R$,there exists $b \in R$ such that $a = a^2 b$.

4) Z,Y,J(R) are respectively the left singular ideal right singular ideal and the Jacobson radical of R .
5) A ring R is said to be semi-commutative if xy=0 implies that xRy=0,for all $x,y \in R$ .It is easy to see that R is semi-commutative if and only if every right (left) annihilator in R is a two-sided ideal [8]

2-MEP-Rings:

In this section we introduce the notion of a right **MEP-ring** with some of their basic properties;

Definition 2.1:

A ring **R** is said to be **right MEP-ring** if every maximal essential right ideal of **R** is left pure.

Next we give the following theorem which plays the key role in several of our proofs.

Theorem 2.2:

Let **R** be a semi commutative, right **MEP–ring**. Then **R** is a reduced ring.

Proof:

Let **a** be a non zero element of **R**, such that $a^2 = 0$ and let **M** be a maximal right ideal containing **r (a)**. We shall prove that **M** is an essential ideal. Suppose that **M** is not essential, then **M** is a direct summand, and hence there exists $0 \neq e = e^2 \in R$ such that **M = r (e) (Lemma 2-3, of [8])**. Since R is semi commutative and $a \in M$, then $ea = 0$ and this implies that $e \in r(a) \subseteq M = r(e)$.

Therefore **e=0**, is a contradiction. Thus **M** is an essential right ideal. Since **R** is a right **MEP- ring**, then **M** is left pure for every $a \in M$**.** Hence there exists $b \in M$ such that **a = ba** implies that $(1- b) \in l(a) = r(a) \subseteq M$, so $1 \in M$ and this implies that **M=R**, which is a contradiction. Therefore **a= 0** and hence **R** is a reduced ring.❒

Theorem 2.3:

If **R** is a semi commutative**,** right **MEP-ring**, then every essential right ideal of **R** is an idempotent.

Proof:

Let **I = bR** be an essential right ideal of **R .** For any element **b** ∈ **I**, **RbR+ r (b)** is essential in **R (Proposition 3 of [5])**.

If **RbR + r (b)** ≠ **R,** let **M** be a maximal right ideal containing **RbR + r(b)** . Since **R** is **MEP – ring** , then there exists **a** ∈ **M** such that **b = ab** and **(1-a)** ∈ **l(b) = r (b)** ⊂ **M**. So **1** ∈ **M** is a contradiction **.**

Thus **RbR + r (b) = R** , and **1 = u + d ,** **u** ∈ **RbR** ⊆ **I ,** **d** ∈ **r (b)** . Hence **b = bu**. Therefore **I =** I^2 (Lemma 3 of [7]). ❐

Proposition 2.4:

Let **R** be a semi commutative, right **MEP-ring**. Then the **J (R) =(0).**

Proof:

Let **0≠a**∈ **J(R)**. If **aR+r (a)** ≠ **R**, then there exists a maximal right ideal **M** containing **aR +r (a)**. Since **a** ∈ **M** and **r (a)** ⊂ **M**, then by a similar method of proof used in Theorem (2.2) **M** is an essential ideal . Since **R** is **MEP – ring** , then there exists **b** ∈ **M**, such that **a = ba** , but **a**∈**J(R)**⊂**M** so **1**∈**M, is** a contradiction. Therefore **aR + r(a) = R** (Proposition 5 of [8]) and **ar +d = 1** , for some **r** ∈ **R** and **d** ∈ **r(a)** , this implies that $a = a^2r$.

Since **a** ∈**J**, then there exists an invertible element **v** in **R** such that **(1-ar) v = 1** , so $(a-a^2r)\, v = a$, yields **a = 0 .** This proves that **J(R) =(0).** ❐

Recall that a ring **R** is said to be **MERT-ring** [7], if every maximal essential right ideal of **R** is a two-sided ideal.

Theorem 2.5:

If **R** is **MERT**, **MEP-ring**, then **Y(R) = (0**).

Proof:

If $Y(R) \neq 0$, by Lemma (7) of [6] , there exists $0 \neq y \in Y(R)$ with $y^2 = 0$. Let **L** be a maximal right ideal of **R**, containing **r(y)** .We claim that **L** is an essential right ideal of **R**.

Suppose this is not true, then there exists a non-zero ideal **T** of **R** such that **L** ⋂**T = (0)** . Then **yRT** ⊆ **LT**⊆ **L** ⋂ **T = 0** impolies **T** ⊆ **r(y)** ⊆ **L,** so **L** ⋂ **T ≠ 0.** This contradiction proves that **L** is an essential right ideal. Since R is an **MEP-ring**, then **L** is a left pure.

Thus for every **y** ∈ **L**, there exists **c** ∈ **L** such that **y = cy (L is a left pure)**. Since **R** is **MERT** , then **cy** ∈**L** (two sided ideal)and thus **1**∈**L**, is a contradiction. Therefore **Y (R) =(0)**.❒

3- The connection between MEP-Rings and other rings

In this section, we study the connection between MEP-Rings and strongly regular rings, weakly regular rings.

Following [3],a ring **R** is **right (left) weakly regular** if $I^2 = I$ for each right (left) ideal **I** of **R**. Equivalently, **a** ∈ **aRaR (a** ∈ **RaRa)** for every **a** ∈ **R** . **R** is **weakly regular** if it's both right and left weakly regular.

The following result is given in [3]:

Lemma 3.1:

A reduced ring **R** is right weakly regular if and only if it is left weakly regular.

Next we give the following lemma:

Lemma 3.2:

If **R a** semi-commutative ring then **RaR+r(a)** is an essential right ideal of **R** for any **a** in **R**.

Proof:

Given **0≠ a** ∈ **R**, assume that **[RaR + r (a)]** ⋂ **I = 0** ,where **I** is a right ideal of **R**. Then **I a** ⊆ **I** ⋂ **RaR = 0,** and so **I** ⊆ **l(a) = r (a)** (**R** is semi commutative).Hence **I = 0**; whence **RaR +r (a)** is an essential right ideal of **R**.❒

Theorem 3.3:

If **R** is a semi commutative, right **MEP-ring**, then **R** is a reduced weakly regular ring.

Proof:

By Theorem (2.2), **R** is a reduced ring .We show that **RaR+r(a)=R**, for any **a** ∈ **R.**

Suppose that **RaR + r (a)** ≠ **R**, then there exists a maximal right ideal **M** containing **RaR + r(a)**.By a similar method of proof used in Theorem (2.2), **M** is an essential ideal.

Now **R** is **MEP- ring**, so **a = ba** , for some **b** ∈ **M** , hence **(1-b)** ∈ **l (a) = r (a)** ⊆ **M** and so **1** ∈ **M** which is a contradiction. Therefore **M=R** and hence **RaR + r (a) = R**, for any **a** ∈ **R**. In particular **1= cab + d, for some c, b** ∈ **R, d** ∈ **r (a)**.

Hence **a = acab** and **R** is right weakly regular. Since **R** is reduced, then by Lemma (3.1) **R** is a weakly regular ring. ❒

Before closing this section, we give the following result.

Theorem 3.4:

A ring **R** is strongly regular if and only if **R** is a semi-commutative, **MEP**, **MERT**- ring.

Proof:

Assume that **R** is **MEP**, **MERT-ring**, let **0** ≠ **a** ∈ **R**, we shall prove that **aR + r (a) = R** . If **aR + r(a)** ≠ **R** , then there exists a maximal right ideal **M** containing **aR + r(a)** . Since **M** is essential, then **M** is left pure. Hence **a= ba** , for some **b** ∈ **M** , so **1** ∈ **M**, a contradiction . Therefore **M=R** and hence **aR+r(a) = R** . In particular **ar +d = 1**, for some **r** ∈ **R, d** ∈ **r(a)**. So $\mathbf{a=a^2r}$.Therefore **R** is strongly regular.

Conversely: Assume that **R** is strongly regular, then by [3], **R** is regular and reduced .Also **R** is **MEP** and semi-commutative.❒

REFERENCES

[1] Al- Ezeh, H. (1989) "Pure ideals in commutative reduced Gelfand rings with unity", **Arch .Math.**, Vol. 53 , PP. 266 –269 .

[2] Al- Ezeh, H. (1989) "On generalized PF-rings", **Math J. Okayama Univ**. Vol.31, PP. 25 – 29 .

[3] Ibraheem, Z. M. (1991) "On P-injective Modules", M. Sc. Thesis, **Mosul University**.

[4] Ming, R. Y. C. (1974) "On simple P-injective modules", **Math. Japonica**, 19, PP. 173-179.

[5] Ming, R. Y. C. (1976) "On VonNumann regular rings-II", **Math.Scand**. 39, PP. 167-170.

[6] Ming, R.Y.C. (1983) "On quasi – injectivity and Von-Neumann regularity" ,Monatash. , **Math . 95**, PP. 25 – 32.

[7] Ming, R.Y.C. (1983) "Maximal ideals in regular rings", **Hokkaido Math. J.** ,Vol. 12, PP. 119-128.

[8] Shuker, N. H. (2004) "On rings whose simple singular modules are GP-injective", **Raf. J. Sci.** ,Vol. 15, No. 1, PP. 37 – 40.