On α-strongly θ-continuity, αθ-openness and (α, θ)-closed graphs in Topological Spaces

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ABSTRACT

Chae   et. al. (1995) have studied the concept of α-strongly θ-continuous functions and (α, θ)-closed graph. The aim of this paper is to investigate several new characterizations and properties of α-strongly θ-continuous functions and   (α, θ)-closed graph. Also, we define a new type of functions called αθ-open functions, which is stronger than quasi α-open and hence strongly α-open, and we obtain some characterizations and properties for it. It is shown that the graph of f, G (f) is (α, θ)-closed graph if and only if for each filter base Ψ in X   θ-converging to some p in X such that f (Ψ) α-converges to some q in Y holds, f (p) = q.

Keywords: α-strongly θ-continuity, αθ-open function and (α,θ)-closed graph.

حول الاستمرارية–θ بقوة–α والمفتوحة–αθ والرسم البياني المغلق–(α, θ) في الفضاءات التبولوجية

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الملخص

دَرَسَ Chae وأخرون سنة 1995 مفهومين α-strongly θ-continuous function و (α, θ)-closed graph. الهدف من هذا البحث هو دراسة مجموعة من المميزات و الخواص للمفهومين. و كذلك تعريف نوع جديد من الدالة المفتوحة من النمط θ- α ، وتكون هذه الدالة أقوى من الدالة من النمط quasi α -open و من ثم أقوى من strongly α-open ، وهكذا حصلنا على مجموعة من الميزات و الخواص لهذه الدالة. وهذا يوضح أن الرسم البياني لدالة f هو (α, θ)-closed graph إذا وفقط إذا كان لكل fiter base Ψ في فضاء X متقاربة من

النمط θ إلى بعض نقاط p في X بحيث أن (Ψ) f متقاربة من النمط α إلى بعض نقاط q في Y يحقق f (p) = q .

الكلمات المفتاحية: الفضاءات التبولوجية، الاستمرارية−θ بقوة−α، دالة مفتوحة−αθ، الرسم البياني المغلق−(α,θ).

1. Introduction

Njastad (1965) introduced and investigated the concept of α-open sets. Chae *et.al.* (1995) have studied the concept of α-strongly θ-continuous functions. It is shown in Chae *et. al.* (1995) that the type of α-strongly θ-continuous function is stronger than a strongly θ-continuous function [25] and a strongly α-continuous function [12].

The purpose of the present paper is to investigate
i) Further characterizations and properties of α-strongly θ-continuous functions [7] and (α, θ)-closed graph [7].
ii) We define a new type of functions called αθ-open functions, which is stronger than quasi α-open and hence strongly α-open, and we obtain some characterizations and properties for it.

2. Preliminaries

Throughout the present paper, spaces always mean topological spaces on which no separation axiom is assumed unless explicitly stated. Let E be a subset of a space X. The closure and the interior of E are denoted by Cl(E) and Int(E), respectively. A subset E is said to be regular open (resp. α-open [22] and semi-open [16]) if E = Int(Cl(E)) (resp. E ⊂ Int(Cl(Int(E))) and E ⊂ Cl(Int(E))). A subset E is said to be θ-open [34](resp. θ-semi-open [26]) if for each x∈E, there exists an open (resp. semi-open) set U in X such that x ∈ U ⊂ Cl(U) ⊂ E. The complement of each regular open (resp. α-open, semi-open, θ-open and θ-semi-open) set is called regular closed (resp. α-closed, semi-closed, θ-closed, θ-semi-closed). The set αCl(E) = { p∈X: E ∩ H ≠ φ for each α-open set H containing p}. A filter base Ψ is said to be θ-convergent [34](resp. α-convergent [32]) to a point x ∈ X if for each open (resp. α-open) set G containing x, there exists an F ∈Ψ such that F ⊂ Cl(G) (resp. F ⊂ G).

In [9], E is a feebly open set in X if there exists an open set U such that U ⊂ E ⊂ sCl(U), where sCl is the semi-closure operator. It is shown in [13] that a set is α-open if and only if it is feebly open. It is well-known that

for a space (X, τ), X can be retopologized by the family τ^α of all α-open sets of X [19] and also the family τ^θ of all θ-open set of X [34], that is, τ^θ (called θ-topology) and τ^α (called an α- topology) are topologies on X, and it is obvious that $\tau^\theta \subset \tau \subset \tau^\alpha$. The family of all α-open (resp. θ-open and feebly-open) set of X is denoted by αO(X) (resp. θO(X) and FO(X)).

• f : X→Y is called strongly θ-continuous [27] if for each x∈X and each open set H of Y containing f(x), there exists an open set G of X containing x such that f(Cl(G)) ⊂ H.

• f : X→Y is called strongly θ-continuous [27] if for each open set H of Y, f^{-1}(H) is θ-open in X if and only if each closed set F of Y, f^{-1}(F) is θ-closed in X.

• f : X→Y is called α-continuous [20] (resp. faintly continuous [18], completely α-irresolute [21] and strongly α-irresolute [12]) if for each open (resp. θ-open, α-open and α-open) set H of Y, f^{-1} (H) is α-open (resp. open, regular open and open) in X.

• f : X→Y is called semi-open [23] (resp. α-open [20], quasi α-open [33], θs-open[1], weakly θs-open[1] and s**-open[2]) function if the image of each open (resp. open, α-open, open, θ-open and semi-open) set G of X, f (G) is semi-open (resp. α-open, open, θ-semi-open, θ-semi-open and open) in Y.

• f : X→Y is called pre-feebly-open[8](resp. strongly α-open [33] and α**-open[2]) function if the image of each α-open set G of X, f (G) is α-open in Y.

It is clear that pre-feebly-open strongly α-open and α**-open functions are equivalent.

• A subset N of a space X is said to be a θ-neighborhood[5] of a point x in X if there exists an open set G such that x ∈ G ⊂ Cl(G) ⊂ N.

• f : X→Y is called θ-open function [5] if for each x ∈ X and each θ-neighborhood N of x, f (N) is θ-neighborhood of f(x).

• A space X is said to be almost regular[31] if for each regularly closed set R of X and each point x ∉ R, there exist disjoint open sets U and V such that R ⊂ U and x∈V.

• A space X is said to be α-Hausdorff [12] if for any x, y ∈ X, x ≠ y, there exist α-open sets G and H such that x ∈ G, y ∈ H and G ∩ H = ∅. It is clear that α-Hausdorff and Hausdorff are equivalent.

- A space X is said to be θ-compact [30] (resp. α-compact [14]) if and only if every cover of X by θ-open (resp. α-open) sets has a finite subcover.
- A subset S of a space X is said to be quasi H-closed [28] relative to X if each cover $\{E_i : i \in I\}$ of S by open sets of X, there exists a finite subset I_0 of I such that $S \subset \cup \{Cl(E_i) : i \in I_0\}$.
- A space X is said to be quasi H-closed [28] if X is quasi H-closed relative to X.
- A function $f : X \to Y$ is said to have θ-closed[24](resp. s*-closed[17], semi-closed[11], θs-closed[1], almost strongly θs-closed[1] and strongly θs-closed[1]) graph if and only if for each $x \in X$ and each $y \in Y$ such that $y \neq f(x)$, there exists an open (resp. semi-open, semi-open, semi-open, semi-open and semi-open) U containing x in X and an open (resp. open, semi-open, open, open and open) set V containing f (x) in Y such that: $(Cl(U) \times Cl(V)) \cap G(f) = \phi$ {resp. $(U \times V) \cap G(f) = \phi$, $(U \times V) \cap G(f) = \phi$, $(Cl(U) \times V) \cap G(f) = \phi$, $(Cl(U) \times Int(Cl(V))) \cap G(f) = \phi$ and $(Cl(U) \times Cl(V)) \cap G(f) = \phi$}.

3. α-strongly θ-continuity

Definition 3.1. A function $f : X \to Y$ is said to be α-strongly θ-continuous [7] if for each $x \in X$ and each α-open set H of Y containing f (x), there exists an open set U of X containing x such that $f(Cl(U)) \subset H$.

The proof of the following theorem is not hard and therefore, it is omitted.

Theorem 3.1. For a function $f : (X, \tau) \to (Y, \gamma)$, the following are equivalent:

i) f is α-strongly θ-continuous;

ii) $f : (X, \tau^\theta) \to (Y, \gamma)$ is strongly α-irresolute;

iii) For each point $x \in X$ and each filterbase Ψ in X θ-converging to x, the filterbase $f(\Psi)$ converges to f (x) in (Y, αO(Y));

iv) For each point $x \in X$ and each net $\{x_\lambda\}_{\lambda \in \nabla}$ in X θ-converging to x, the net $\{f\{x_\lambda\}\}_{\lambda \in \nabla}$ converges to f (x) in (Y, αO(Y));

v) For each point $x \in X$ and each filterable Ψ in X θ-converging to x, the fiterbase $f(\Psi)$ α-converges to f (x) in (Y, γ);

vi) For each point $x \in X$ and each net $\{x_\lambda\}_{\lambda \in \nabla}$ in X θ-converging to x, the net $\{f\{x_\lambda\}\}_{\lambda \in \nabla}$ α-converges to f (x) in (Y, γ).

Lemma 3.1. (Andrijevic [4]). Let E be a subset of a space (X, τ). Then the following hold.

1) $\alpha Cl(E) = E \cup Cl(Int(Cl(E)))$;

2) $\alpha Int(E) = E \cap Int(Cl(Int(E)))$.

Theorem 3.2. For $f: X \rightarrow Y$, the following are equivalent:

a) f is α-strongly θ-continuous;

b) $f(Cl_\theta(E)) \subset f(E) \cap Cl(Int(Cl(f(E))))$, for each subset E of X;

c) $Cl_\theta(f^{-1}(W)) \subset f^{-1}(W \cup Cl(Int(Cl(W))))$, for each subset W of Y;

d) $f^{-1}(W \cap Int(Cl(Int(W)))) \subset Int_\theta(f^{-1}(W))$, for each subset W of Y.

Proof. This follows from Lemma 3.1 and Theorem 2 of [7].

Theorem 3.3. If $f: X \rightarrow Y$ is α-strongly θ-continuous and if E is an open subset of X, then $f|E: E \rightarrow Y$ is α-strongly θ-continuous.

Proof. Let H be any α-open subset of Y. Since f is α-strongly θ-continuous. By [7, Theorem 2], $f^{-1}(H) \in \theta O(X)$, so by Lemma 1.2.9 of [1], $(f|E)^{-1}(H) = f^{-1}(H) \cap E \in \theta O(E)$. This implies that $f|E: E \rightarrow Y$ is α-strongly θ-continuous.

The proof of the following result directly is true.

Theorem 3.4. For any two functions, $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, the following are true:

i) if f is α-strongly θ-continuous and g is α-continuous, then $g \circ f$ is strongly θ-continuous.

ii) if f is faintly continuous and g is α-strongly θ-continuous, then $g \circ f$ is strongly α-irresolute.

Theorem 3.5. Let $f: X \rightarrow Y$ be a function. If $g: Y \rightarrow Z$ is an α-open bijection and $g \circ f: X \rightarrow Z$ is α-strongly θ-continuous, then f is strongly θ-continuous.

Proof. Suppose g is α-open function. Let H be an open subset of Y, since g is one to one and onto, then the set $g(H)$ is an α-open subset of Z, since $g \circ f$ is α-strongly   θ-continuous, it follows that $(g \circ f)^{-1}(g(H)) = f^{-1}(g^{-1}(g(H))) = f^{-1}(H)$ is θ-open in X. Thus, f is strongly θ-continuous.

Theorem 3.6. If X is almost regular and $f : X \to Y$ is completely α-irresolute function, then f is α-strongly θ-continuous.

Proof. Let H be an α-open subset of Y. Since f is completely α-irresolute, $f^{-1}(H)$ is regular open in X and from the fact that a space X is almost regular if and only if for each $x \in X$ and each regular open set $f^{-1}(H)$ containing x, there exists a regular open set O such that $x \in O \subset Cl(O) \subset f^{-1}(H)$ [31, Theorem 2.2]. Therefore, $f^{-1}(H)$ is θ-open in X and by [7, Theorem 2], f is α-strongly θ-continuous.

Lemma 3.2 [10]. Let $\{ X_\lambda : \lambda \in \Delta \}$ be a family of spaces and $U_{\lambda i}$ be a subset of $X_{\lambda i}$ for each i=1,2,…,n. Then $U = \prod_{i=1}^{n} U_{\lambda i} \times \prod_{\lambda \neq \lambda \iota} X_\lambda$ is α-open in $\prod_{\lambda \in \Delta} X_\lambda$ if and only if $U_{\lambda i} \in \alpha O(X_{\lambda i})$ for each i=1,2,…,n.

Theorem 3.7. Let $g_\lambda : X_\lambda \to Y_\lambda$ be a function for each $\lambda \in \Delta$ and $g : \prod X_\lambda \to \prod Y_\lambda$ a function defined by $g(\{ x_\lambda \}) = \{ g_\lambda (x_\lambda) \}$ for each $\{ x_\lambda \} \in \prod X_\lambda$. If g is α-strongly θ-continuous, then g_λ is α-strongly θ-continuous for each $\lambda \in \Delta$.

Proof. Let $\beta \in \Delta$ and $V_\beta \in \alpha O(Y_\beta)$. Then, by Lemma 3.2, $V = V_\beta \times \prod_{\lambda \neq \beta} Y_\lambda$ is α-open in $\prod Y_\lambda$ and $g^{-1}(V) = g^{-1}_\beta (V_\beta) \times \prod_{\lambda \neq \beta} X_\lambda$ is θ-open in $\prod X_\lambda$. From Lemma 3.2, $g^{-1}_\beta (V_\beta) \in \theta O(X)$. Therefore, g_β is α-strongly θ-continuous.

Remark 3.1. It was known in [6, Example 2.2] that $V \in \alpha O(X \times Y)$ may not, generally, be a union of sets of the form $A \times B$ in the product space $X \times Y$, where $A \in \alpha O(X)$ and $B \in \alpha O(Y)$. Therefore, the converse of Theorem 3.8 may not be true, generally.

Theorem 3.8. Let $g : X \to Y_1 \times Y_2$ be α-strongly θ-continuous function and $f_i : X \to Y_i$ be coordinate functions, i.e. for $x \in X$, $g(x) = (x_1, x_2)$, $f_i(x) = x_i$, $i = 1, 2$. Then $f_i : X \to Y_i$ is α-strongly θ-continuous for $i = 1, 2$.

Proof. Let x be any point in X and H_1 be any α-open set in Y_1 containing $f_1(x) = x_1$, then by Lemma 3.2, $H_1 \times Y_2$ is α-open in $Y_1 \times Y_2$, which

contain (x_1, x_2). Since g is α-strongly θ-continuous, there exists an open set U containing x such that $g(Cl(U)) \subset H_1 \times Y_2$. Then $f_1(Cl(U)) \times f_2(Cl(U)) \subset H_1 \times Y_2$. Therefore, $f_1(Cl(U)) \subset H_1$. Hence f_1 is α-strongly θ-continuous.

Lemma 3.3. Let $X_1, X_2, \ldots, X_n$ be n topological spaces and $X = \prod_{i=1}^{n} X_i$.

Let $E_i \in \theta O(X_i)$, for $i = 1, 2, \ldots, n$. Then $\prod_{i=1}^{n} E_i \in \theta O(\prod_{i=1}^{n} X_i)$.

Proof. Let $(x_1, x_2, \ldots, x_n) \in \prod_{i=1}^{n} E_i$, then $x_i \in E_i$, for $i = 1, 2, \ldots, n$. Since $E_i \in \theta O(X_i)$, for $i = 1, 2, \ldots, n$. Then, there exist open sets U_i, for $i = 1, 2, \ldots, n$ such that $x_i \in U_i \subset Cl(U_i) \subset E_i$, for $i = 1, 2, \ldots, n$. Therefore, $(x_1, x_2, \ldots, x_n) \in U_1 \times U_2 \times \ldots \times U_n \subset Cl(U_1) \times Cl(U_2) \times \ldots \times Cl(U_n) = Cl_{X_1 \times X_2 \times \ldots \times X_n}(U_1 \times U_2 \times \ldots \times U_n) \subset \prod_{i=1}^{n} E_i$ and $\prod_{i=1}^{n} U_i \in \tau(\prod_{i=1}^{n} X_i)$. Hence $\prod_{i=1}^{n} E_i$ is θ-open set in $\prod_{i=1}^{n} X_i$.

Theorem 3.9. Let $X_1, X_2, \ldots, X_n$ and Z be topological spaces and $f : \prod_{i=1}^{n} X_i \to Z$. If given any point p of $\prod_{i=1}^{n} X_i$, and given any α-open set U in Z containing $f(p)$, there exist θ-open sets E_i in X_i for $i = 1, 2, \ldots, n$ such that $p \in \prod_{i=1}^{n} E_i$ and $f(\prod_{i=1}^{n} E_i) \subset U$. Then f is α-strongly θ-continuous.

Proof. Let $p \in \prod_{i=1}^{n} X_i$ and U be any α-open set in Z containing $f(p)$. By hypothesis, there exist θ-open sets E_i in X_i for $i = 1, 2, \ldots, n$ such that $p \in$

$\prod\limits_{i=1}^{n} E_i$ and $f(\prod\limits_{i=1}^{n} E_i) \subset U$. Since $E_i \in \theta O(X_i)$, for $i = 1, 2, \ldots, n$.

Therefore, by Lemma 3.3, $\prod\limits_{i=1}^{n} E_i \in \theta O(\prod\limits_{i=1}^{n} X_i)$, for $i = 1, 2, \ldots, n$. Thus, f

is α-strongly θ-continuous.

4. $\alpha\theta$-open Functions.

In this section we define a new type of functions called $\alpha\theta$-open function and we find some characterization and properties for it.

Definition 4.1. A function $f : X \rightarrow Y$ is called $\alpha\theta$-open if and only if for each α-open set G in X, $f(G) \in \theta O(Y)$.

It follows immediately that every $\alpha\theta$-open functions is quasi α-open and hence strongly α-open, the converse is not true as seen from the following example.

Example 4.1. Let X = {a, b, c, d} and τ = { X, ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}}. The identity function $i : (X, \tau) \rightarrow (X, \tau)$ is strongly α-open, but it is not $\alpha\theta$-open function since {a} $\in \alpha O(X, \tau)$, but $f(\{a\}) = \{a\}$ $\notin \theta O(X, \tau)$.

We find some characterizations and properties of $\alpha\theta$-open functions.

Theorem 4.1. For any bijection function $f : X \rightarrow Y$, the following are equivalent:
i) The inverse function $f^{-1}: Y \rightarrow X$ is α-strongly θ-continuous;
ii) $f: X \rightarrow Y$ is $\alpha\theta$-open function.
Proof. Follows from their definitions.

Theorem 4.2. For a function $f : X \rightarrow Y$, the following are equivalent:
a) f is $\alpha\theta$-open function;
b) $f(\alpha Int(E)) \subset Int_\theta(f(E))$, for each subset E of X;
c) $\alpha Int(f^{-1}(W)) \subset f^{-1}(Int_\theta(W))$, for each subset W of Y;
d) $f^{-1}(Cl_\theta(W)) \subset \alpha Cl(f^{-1}(W))$, for each subset W of Y.
Proof. (a)$\Rightarrow$(b). Suppose f is $\alpha\theta$-open function and E $\subset$ X. Since $\alpha Int(E) \subset$ E, $f(\alpha Int(E)) \in \theta O(Y)$ and $f(\alpha Int(E)) \subset f(E)$ and hence $f(\alpha Int(E)) \subset Int_\theta(f(E))$.

(b)$\Rightarrow$(c). Let $W \subset Y$. Then $f^{-1}(W) \subset X$. Therefore, we apply (b), we obtain $f(\alpha\text{Int}(f^{-1}(W))) \subset \text{Int}_\theta(f(f^{-1}(W)))$. Then, $\alpha\text{Int}(f^{-1}(W)) \subset f^{-1}(\text{Int}_\theta(W))$.

(c)$\Rightarrow$(d). Let $W \subset Y$, then apply (c) to $Y \setminus W$, we get $\alpha\text{Int}(f^{-1}(Y \setminus W)) \subset f^{-1}(\text{Int}_\theta(Y \setminus W))$. Then, $\alpha\text{Int}(X \setminus f^{-1}(W)) \subset f^{-1}(Y \setminus \text{Cl}_\theta(W))$, which implies that $X \setminus \alpha\text{Cl}(f^{-1}(W)) \subset X \setminus f^{-1}(\text{Cl}_\theta(W))$. Hence $f^{-1}(\text{Cl}_\theta(W)) \subset \alpha\text{Cl}(f^{-1}(W))$.

(d)$\Rightarrow$(a). Let G be any α-open set in X. Then $Y \setminus f(G) \subset Y$, apply (d), we obtain $f^{-1}(\text{Cl}_\theta(Y \setminus f(G))) \subset \alpha\text{Cl}(f^{-1}(Y \setminus f(G)))$. Then $f^{-1}(Y \setminus \text{Int}_\theta(f(G))) \subset \alpha\text{Cl}(X \setminus G)$. Which implies that $X \setminus f^{-1}(\text{Int}_\theta(f(G))) \subset X \setminus \text{Int } G = X \setminus G$. Therefore, $G \subset f^{-1}(\text{Int}_\theta(f(G)))$. Then, $f(G) \subset \text{Int}_\theta(f(G))$. Therefore, $f(G) \in \theta O(Y)$. which completes the proof.

Remark 4.1. Let $f : X \rightarrow Y$ be a bijective function. Then, f is $\alpha\theta$-open function if and only if $f(F) \in \theta C(Y)$, for each α-closed set F in X.

Theorem 4.3. If Y is a regular space, then each s**-open function is $\alpha\theta$-open.

Proof. Let G be any α-open subset of X, then it is semi-open. Since f is s**-open function. Therefore, $f(G)$ is open in Y. But Y is a regular space, then by [1, Lemma 1.2.8] $f(G)$ is θ-open in Y.

Theorem 4.4. If a function $f : X \rightarrow Y$ is $\alpha\theta$-open and $E \subset X$ is an open set in X, then the restriction $f|E: E \rightarrow Y$ is $\alpha\theta$-open function.

Proof. Let H be any α-open set in the open subspace E. Then, by [15, Theorem 3.7], H is α-open in X. Since f is $\alpha\theta$-open function. Therefore, $f(H)$ is θ-open in Y. Hence $f|E$ is $\alpha\theta$-open function.

Theorem 4.5. Let $f : X \rightarrow Y$ be a function and $\{E_\alpha : \alpha \in \nabla\}$ be an open cover of X. If the restriction $f|E_\alpha : E_\alpha \rightarrow Y$ is $\alpha\theta$-open function for each $\alpha \in \nabla$, then f is $\alpha\theta$-open function.

Proof. Let H be any α-open set in X. Therefore, by [15, Theorem 3.4], $H \cap E_\alpha$ is α-open in the subspace E_α for each $\alpha \in \nabla$. Since $f|E_\alpha$ is $\alpha\theta$-open function $(f|E_\alpha)(H \cap E_\alpha)$ is θ-open in Y and hence $f(H) = \cup\{(f|E_\alpha)(H \cap E_\alpha) : \alpha \in \nabla\}$ is θ-open in Y. This shows that f is $\alpha\theta$-open function.

Theorem 4.6. A function $f : X \rightarrow Y$ is $\alpha\theta$-open if and only if for each subset S of Y and any α-closed set F in X containing $f^{-1}(S)$, there exists a θ-closed set M in Y containing S such that $f^{-1}(M) \subset F$.

Proof. Suppose that f is $\alpha\theta$-open function. Let $S \subset Y$ and F be an α-closed set in X containing $f^{-1}(S)$. Put $M = Y \setminus f(X \setminus F)$, then M is θ-closed in Y and since $f^{-1}(S) \subset F$, we have $S \subset M$. Since f is $\alpha\theta$-open function and F is α-closed in X, M is θ-closed in Y. Obviously $f^{-1}(M) \subset F$.

Conversely, let G be any α-open subset of X and put $S = Y \setminus f(G)$. Then, $X \setminus G$ is α-closed set containing $f^{-1}(S)$. By hypothesis, there exists a θ-closed set M in Y containing S such that $f^{-1}(M) \subset X \setminus G$. Thus, we have $f(G) \subset Y \setminus M$. On the other hand, we have $f(G) = Y \setminus S \supset Y \setminus M$ and hence $f(G) = Y \setminus M$. Consequently, $f(G)$ is θ-open in Y and f is $\alpha\theta$-open function.

5. Functions with (α, θ)-closed graph

In this section we investigate several new properties of (α, θ)-closed graph [7].

Definition 5.1[7]. Let $G(f) = \{(x, f(x)): x \in X\}$ be the graph of $f : X \rightarrow Y$ then $G(f)$ is said to be (α, θ)-closed with respect to $X \times Y$, if for each point $(x, y) \notin G(f)$, there exists an open set U and an α-open set H containing x and y, respectively such that $f(Cl(U)) \cap H = \phi$.

The following diagram is an enlargement of the diagram 4.1.1 of [1]. Note that none of the implications is reversible
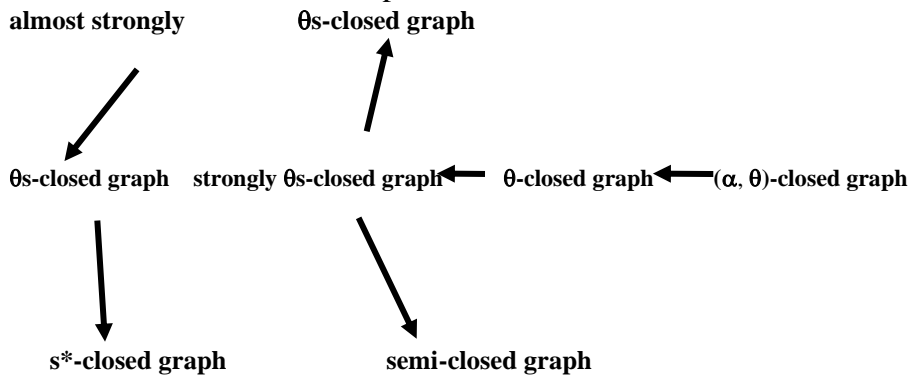

Diagram 5.1

Example 5.1. Let X = {a, b, c } and τ = {φ, X, {a}, {a, b},{a, c}}, then the function $f : (X, τ) \to (X, τ)$ is defined as:
$f(x) = a$, for each x ∈ X, has θ-closed graph, which has not (α, θ)-closed graph.

Theorem 5.1. If $f : X \to Z$ is a function with (α, θ)-closed graph, and $g : Y \to Z$ is α-strongly θ-continuous functions, then the set {(x, y) : $f(x) = g(y)$} is θ-closed in X × Y.

Proof. Let E = {(x, y): $f(x) = g(y)$}. If (x, y) ∈ X × Y \ E, then $f(x) \neq g(y)$. Hence (x, g(y)) ∈ (X × Z) \ G(f). Since f has (α, θ)-closed graph. Therefore, there exists open set U ⊂ X and α-open set H ⊂ Z containing x and g(y), respectively, such that $f(Cl(U)) \cap H = \phi$. The α-strongly θ-continuity of g implies that there is an open set V of Y such that g(Cl(V)) ⊂ H. Therefore, we have $f(Cl(U)) \cap g(Cl(V)) = \phi$. This establishes that (Cl(U) × Cl(V)) ∩ E = φ, which implies that (x, y) ∉ Cl$_θ$(E). So, E is θ-closed in X × Y.

Corollary 5.1. If Y is an Hausdorff space and f, g: X→Y are α-strongly θ-continuous function, then the set {(x, y) : $f(x) = g(y)$} is θ-closed in X × X.

Proof. Follows from Theorem 5.1 and Theorem 16 of [7].

Theorem 5.2. If $f : X \to Y$ is any function with θ-closed point inverses such that the image of closure of each open set is α-closed, then f has (α, θ)-closed graph.

Proof. Let (x, y) ∈ (X × Y) \ G(f). Then x ∉ $f^{-1}(y)$ and since $f^{-1}(y)$ is θ-closed, there exists an open set U containing x such that Cl(U) ∩ $f^{-1}(y) = \phi$. By assumption $f(Cl(U))$ is α-closed. Since y ∉ $f(Cl(U))$, there is an α-open set H in Y containing y such that $f(Cl(U)) \cap H = \phi$. Thus f has (α, θ)-closed graph.

Theorem 5.3. Let $f : X \to Y$ be a function with (α, θ)-closed graph, then for each x ∈ X, {$f(x)$} = ∩ {αCl($f(Cl(U))$): U is an open set containing x}

Proof. Let the graph of the function be (α, θ)-closed. If {$f(x)$} ≠ ∩ {αCl($f(Cl(U))$): U is an open set containing x}.

Let $y \neq f(x)$ such that $y \in \cap \{\alpha Cl(f(Cl(U)))$: U is an open set containing x}. This implies that $y \in \alpha Cl(f(Cl(U)))$ for each open set containing x; it means that , for each α-open set V containing y in Y, $V \cap f(Cl(U)) \neq \phi$. That contradicts Definition 5.1. Thus $y = f(x)$.

Theorem 5.4. Let $f : X \to Y$ be a function with (α, θ)-closed graph. If E is quasi H-closed in X, then $f(E)$ is α-closed in Y.

Proof. Let E be a quasi H-closed in X. Suppose that $f(E)$ is not α-closed in Y. Let $y \notin f(E)$. Therefore, $y \neq f(x)$ for each $x \in E$. Since $G(f)$ has (α, θ)-closed, for each $x \in E$, there exists open set U_x and α-open set H_x containing x and y, respectively such that $f(Cl(U_x)) \cap H_x = \phi$, for each $x \in E$. The family $\mathbf{Q} = \{U_x : x \in E\}$ is an open cover of E. Since E is quasi H-closed, there exists a finite subfamily $\{U_{x(1)},\ldots, U_{x(n)}\}$ of $\mathbf{Q}$ such that

$E \subset \bigcup\limits_{i=1}^{n} Cl(U_{x(i)})$. Put

$H = \bigcap\limits_{i=1}^{n} H_{x(i)}$. Then, $f(E) \cap H \subset \bigcup\limits_{i=1}^{n} (f(Cl(U_{x(i)}))) \cap H \subset \bigcup\limits_{i=1}^{n} (f(Cl(U_{x(i)})) \cap H_{x(i)}) = \phi$.

Since H is an α-open set containing y, $y \notin \alpha Cl(f(E))$. Therefore, $\alpha Cl(f(E)) \subset f(E)$, which implies that $f(E)$ is α-closed.

Corollary 5.2. The image of any quasi H-closed space in any space is α-closed under functions with (α, θ)-closed graphs.

Theorem 5.5. Let $f : X \to Y$ be given. Then $G(f)$ is (α, θ)-closed graph if and only if for each filter base Ψ in X θ-converging to some p in X such that $f(\Psi)$ α-converges to some q in Y holds, $f(p) = q$.

Proof. Suppose that $G(f)$ is (α, θ)-closed graph and let $\Psi = \{E_\delta : \delta \in \nabla\}$ be a filter base in X such that Ψ θ-converges to p and $f(\Psi)$ α-converges to q . If $f(p) \neq q$, then $(p, q) \notin G(f)$. Thus, there exists an open set $U \subset X$ and α-open set $V \subset Y$ containing p and q, respectively, such that $(Cl(U) \times V) \cap G(f) = \phi$. Since Ψ θ-converges to p and $f(\Psi)$ α-converges to q, there exists an $E_\delta \in \Psi$ such that $E_\delta \subset Cl(U)$ and $f(E_\delta) \subset V$. Consequently, $(Cl(U) \times V) \cap G(f) \neq \phi$, which is a contradiction.

Conversely, assume that G(f) is not (α, θ)-closed graph. Then, there exists a point (p, q) $\notin$ G (f) such that for each open set U $\subset$ X and each α-open set V $\subset$ Y containing p and q, respectively, holds (Cl(U) $\times$ V) $\cap$ G (f) $\neq$ ϕ. Let { U$_\delta$: δ $\in$ ∇_1}be the set of all open sets of X containing p. Define

Ψ_1 = {Cl(U$_\delta$): δ $\in$$\nabla_1$},

Ψ_2 = {V$_\beta$: V$_\beta$ is an α-open set containing q and β $\in$$\nabla_2$}

Ψ_3 = {E (δ, β): E (δ, β) = (Cl(U$_\delta$) $\times$ V$_\beta$) $\cap$ G (f), (δ, β) $\in$ $\nabla_1 \times \nabla_2$} and

Ψ = {Ψ* (δ, β): (δ, β) $\in$$\nabla_1 \times \nabla_2$}, where

Ψ* (δ, β) = {x $\in$U$_x$: (x, f(x)) $\in$ E (δ, β)}. Then Ψ is a filter base in X with property that Ψ θ-converges to p, f (Ψ) α-converges to q , and f(p) $\neq$ q . This completes the proof.

Corollary 5.3. A function f : X$\rightarrow$ Y has (α, θ)-closed graph if and only if for each net {x$_\gamma$} in X such that x$_\gamma$ $\rightarrow_\theta$ p $\in$ X and f (x$_\gamma$) $\rightarrow_\alpha$ q $\in$ Y holds, f (p) = q.

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