On Dual Rings

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ABSTRACT

A ring R is called a right dual ring if rl(T) = T for all right ideals T of R. The main purpose of this paper is to develop some basic properties of dual rings and to give the connection between dual rings, regular rings and strongly regular rings.

Keywords: dual rings, regular rings, strongly regular rings.

حول الحلقات الاثنينية

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الملخص

يقال للحلقة R بأنها اثنينية يمنى إذا كان rl(T)=T لكل مثالي أيمن T في R. الهدف من هذا البحث هو تطوير بعض الخواص الأساسية للحلقات الاثنينية، وإيجاد بعض العلاقات التي تربط الحلقات الاثنينية والحلقات المنتظمة والحلقات المنتظمة بقوة.

الكلمات المفتاحية: الحلقات الاثنينية، الحلقات المنتظمة، الحلقات المنتظمة بقوة.

1. INTRODUCTION

Throughout this paper, R represents an associative ring with identity and all R-modules are unitary. Recall that: (1) A ring R is reduced if R contains no non–zero nilpotent element ; (2) R is said to be von Neumann regular (or just regular) ring if $a \in aRa$ for every a in R; (3) A right R-module M is called P-injective if, for any principal right ideal I of R, every right R-homomorphism of I into M extends to R. we say that, R is a right P-injective ring if R_R is P-injective; (4) R is called right duo-ring if every right ideal of R is a two- sided ideal; (5) For every $a \in R$, r(a) and l(a) will stand respectively for right and left annihilators of a; (6) Y(R) will denote the right singular ideal of R.

2. DUAL RINGS (BASIC PROPERTIES).

Following [7], a ring R is said to be a right dual ring if rl(T)=T, for all right ideals T of R. A left dual ring is similarly defined . 1

A ring R is called dual ring if R is a right and left dual ring.

Example. Let R be the set of all 2 × 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with a, b, c, d $\in Z_2$ (The ring of integers modulo 2).

A straightforward calculation, shows that R is a dual ring.

Following [1], a ring R is said to be a right Ikeda-Nakayama ring (right IN- ring) if $l(A \cap B) = l(A) + l(B)$ for all right ideals A and B of R.

In [3], Hajarnavis and Norton Proved that:

Lemma 2.1. Every dual ring is IN- ring.

We begin this section with the following lemma.

Lemma 2.2. Let R be a right dual ring, and let M_1 and M_2 be right ideals of R. Then $M_1 \subseteq M_2$ if and only if $l(M_2) \subseteq l(M_1)$.

Proof.
If $M_1 \subseteq M_2$, then obviously $l(M_2) \subseteq l(M_1)$.
Conversely, assume that $l(M_2) \subseteq l(M_1)$. Then $rl(M_1) \subseteq rl(M_2)$. By duality of R, we have $M_1 \subseteq M_2$.

The next proposition is a direct consequence of Lemma 2.2

Proposition 2.3. Let R be a dual ring. Then

1- M is a maximal right ideal of R if and only if l(M) is minimal left ideal.
2- M is a minimal right ideal of R if and only if l(M) is maximal left ideal.

Proof
(1). Let M be a maximal right ideal of R, and let L be a left ideal of R such that $(0) \subseteq L \subseteq l(M)$. Then by Lemma 2.2 ,
$R=r(0) \supseteq r(L) \supseteq rl(M) = M$. By maximally of M, we have $r(L) = R$, and this implies $L = l(R)$. Therefore $L = (0)$.
Conversely, assume that l(M) is a minimal left ideal of R, and let $M \subset I \subseteq R$ for some right ideal I of R.
Then $l(M) \supset l(I) \supseteq l(R) = (0)$. Hence $l(I) =(0)$, so $I = R$
(2). Let M be a minimal right ideal of R, and let L be a left ideal of R such that $l(M) \subset L \subseteq R$.
Then $M = rl(M) \supset r(L) \supseteq r(R) = (0)$. So $r(L) =(0)$ and hence $L = R$.
Conversely, let l(M) be a maximal left ideal, and let L be a right ideal such that $(0) \subseteq L \subseteq M$, then $R = l(0) \supseteq l(L) \supseteq l(M)$. So $l(L) = R$. Hence $L = (0)$.

Recall the following result of Nicholson and Yousif
[5, Lemma 1.1].

Lemma 2.4. The following conditions are equivalent

1- R_R is P-injective.

2- lr(a)= Ra for all a in R.

3- If $r(b) \subseteq r(a)$, for $a, b \in R$, then $Ra \subseteq Rb$.

4- $l[bR \cap r(a)] = l(b) + Ra$, for all $a, b \in R$.

Theorem 2.5 Let R be a right Noetherian P-injective ring, and let $r(L_1 \cap L_2) = r(L_1)+r(L_2)$ for all principal left ideals L_1 and L_2 of R. Then R is a right dual ring.

Proof. Let $0 \neq a \in R$. First we claim that aR = rl(aR) .
Clearly $aR \subseteq rl(aR)$. Let $b \in rl(aR)$. Then xb = 0 for all $x \in l(aR)$. Since $l(aR) \subseteq l(bR)$,then define $f: Ra \rightarrow Rb$, by f(xa) = xb. Clearly f is a well defined left R – homomorphism.
Since R is P-injective, there exists $c \in R$ such that xb = f(xa) = xac for all $x \in R$, whence $b = ac \in aR$, yielding aR = rl(aR). Since R is right Noetherian, then by [4, Theorem 2.3.13], every right ideal I of R can be written in the form

$I = a_1R+a_2R+\ldots+a_nR$, and this implies

$$rl(I) = r(l(a_1R) \cap l(a_2R) \cap \ldots \cap l(a_nR))$$
$$= rl(a_1R) + rl(a_2R) + \ldots + rl(a_nR)$$
$$= a_1R + a_2R + \ldots + a_nR = I.$$

3.THE CONNECTION BETWEEN DUAL RINGS AND REGULAR RINGS .

The Purpose of this section is to show the connection between dual rings, regular rings and strongly regular rings.

Recall that a ring R is strongly regular if for every $a \in R$, $a \in a^2R$. Clearly a strongly regular ring is a reduced regular ring.
We begin this section with the following result.

Theorem 3.1. Let R be a reduced left or right dual ring. Then R is strongly regular.

Proof.
Let a be a non – zero element in R. Then $r(a) = r(a^2)$ (R is reduced). Since R is a left dual ring, by [6, Theorem 11], R_R is P-injective, and hence $Ra = lr(a)$ (Lemma 2.4). Whence $Ra = lr(a) = lr(a^2) = Ra^2$. This implies that $a = ra^2$, for some $r \in R$. Therefore R is strongly regular.

Next, we give other sufficient condition for dual ring to be strongly regular.

Theorem 3.2. Let R be a semi-prime left dual ring and right duo-ring. Then R is strongly regular.

Proof.
Let $0 \neq a \in R$, and let $I = r(a) \cap aR$, first we claim that $I^2 = (0)$.Suppose that $I^2 \neq (0)$. For any $d \in I$, $d \in r(a)$ and $d \in aR = Ra$ (R is a right duo-ring), so $d = ba$ for some $b \in R$, and $aba=0$. Thus $d^2 =0$ and hence $I^2 = (0)$. Since R is semi-prime, then $I=(0)$. Next, we claim that $r(a) = r(a^2)$, clearly $r(a) \subseteq r(a^2)$. Let $x \in r(a^2)$. Then $a^2x = 0$, so $a (ax) = 0$ and hence $ax \in r(a)$, but $ax \in aR$, then $ax \in aR \cap r(a) = (0)$. Therefore $x \in r(a)$. On the other hand since R is a left dual ring then $Ra = lr(a) = lr(a^2) = Ra^2$. Therefore R is strongly regular.

The next result provides a link between dual rings and regular rings.

Theorem 3.3. Let R be a right non–singular dual ring. Then R is regular ring.

Proof.
Let $0 \neq a \in R$, then by [6. Theorem 11] and (Lemma 2.4), $Ra = lr(a)$. Since R is a right non – singular ring, then $Y(R) = 0$.

Whence r(a) is not essential right ideal of R. Then there exists a non-zero right ideal L of R such that r(a) $\oplus$ L is essential right ideal of R. Now by Lemma 2.1 R is a right IN–ring. Then we have lr(a) +l(L) = l(r(a)$\cap$L)=R. Whence it follows that Ra + l(L) = R, while lr(a) $\cap$l(L) $\subseteq$ l(r(a) +L) = (0). So Ra $\cap$ l(L) = (0).
Thus Ra = lr(a) is a direct summand. Therefore R is regular [2, Theorem 1.1].
Before closing this section we present the following result.

Proposition 3.4. Let R be a regular ring.
Then $r(L_1 \cap L_2) = r(L_1) + r(L_2)$ for all principal left ideals L_1 and L_2 of R.

Proof.
Obviously $r(L_1) + r(L_2) \subseteq r(L_1 \cap L_2)$ always holds.
Let $b \in r(L_1 \cap L_2)$, define $f_i \in Hom_R(L_i, {}_RR)$, i = 1.2 as follows: $f_1(a_1)=a_1$ for all $a_1 \in L_1$ and $f_2(a_2) = a_2(1-b)$ for all $a_2 \in L_2$. The mapping $f(a_1+a_2)=f_1(a_1)+f_2(a_2)$ is a well defined left R-homomorphism,indeed if, $a_1 + a_2 = a'_1 + a'_2$ then $a_1 - a'_1 = -a_2 + a'_2 \in L_1 \cap L_2$ .But $b \in r(L_1 \cap L_2)$ therefore $a_2 b = a'_2 b$. Showing that $f(a_1 + a_2) = f(a'_1 + a'_2)$.Since R is regular, then ${}_RR$ is P-injective, so there exists $c \in R$ such that $f(a_1 + a_2) = (a_1 + a_2)c$.
This implies $a_1 + a_2(1-b) = f(a_1 + a_2) = (a_1 + a_2)c$, and therefore $a_1(1-c) + a_2(1-b-c) = 0$ for all $a_1 \in L_1$ and $a_2 \in L_2$. It follows that $1-c \in r(L_1)$ and $1-b-c \in r(L_2)$.
Therefore $b=(1-c) - (1-b-c) \in r(L_1) + r(L_2)$.
This shows $r(L_1 \cap L_2) = r(L_1)+r(L_2)$ for all principal left ideals L_1 and L_2 of R.

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