An Approximate Solution to The Newell-Whitehead Equation by Adomian Decomposition Method

Saad A. Manaa
College Faculty of Science
University of Zakho, Iraq

Received on: 20 / 1 / 2011 **Accepted on: 4 / 4 / 2011**

ABSTRACT

In this paper, we solved the Newell-Whitehead equation approximately using Adomain Decomposition method and we have compared this solution with the exact solution; we found that the solution of this method is so close to the exact solution and this solution is slower to converge to the exact solution when we increase *t* however, this method is effective for this kind of problems.

KEYWORDS: Partial Differential Equations, Newell-Whitehead equation, Adomain Decomposition method.

الحل التقريبي لمعادلة (Newell - Whitehead) باستخدام طريقة (Adomain Decomposition)

سعد مناع
كلية العلوم، جامعة زاخو

تاريخ استلام البحث: 2011/01/20 **تاريخ قبول البحث: 2011/04/04**

الملخص

تم في هذا البحث إيجاد الحل التقريبي لمعادلة (Newell - Whitehead) باستخدام طريقة (Adomain Decomposition) وتم عمل مقارنة مع الحل المضبوط وتبين أن الحل باستخدام هذه الطريقة قريب جداً من الحل المضبوط ويكون أبطأ في الوصول إلى الحل المضبوط كلما زادت قيمة *t* وان هذه الطريقة فعالة جداً في حل هذا النوع من المسائل.

الكلمات المفتاحية: معادلات تفاضلية جزئية، معادلة Newell-Whitehead، طريقة Adomain Decomposition.

1. Introduction

The decomposition method was first introduced by Adomian since the beginning of the 1980s [7]. The Adomian method [3] can be used for solving a wide range of problems whose mathematical models yield equation or system of equations involving algebraic, differential, integral and integro-differential. In this method the solution is considered as the sum of an infinite series, rapidly converging to an accurate solution.

It is well known that the key of the method is to decompose the nonlinear term in the equations into a peculiar series of polynomials $\sum_{n=1}^{\infty} A_n$, where A_n are the so-called Adomian polynomials [4–6].

This iterative method has been proven to be rather successful in dealing with linear problems as well as nonlinear. Adomian gives the solution as an infinite series usually converging to an accurate solution.

An Analytical Solution of the Stochastic Navier–Stokes System is shown by Adomian [3]. The decomposition method used to solve a system of partial differential equations and in reaction–diffusion to the Brusselator model and finding that the Adomian series solution gives an excellent approximation to the exact solution [20]. Wazwaz [22] developed a fast and accurate algorithm for the solution of sixth-order boundary value problems (BVP) and the modified decomposition method [19, 21]. Abbasbandy [1, 2] and Allan [8] studied some efficient numerical algorithms to solve a system of two nonlinear equations (with two variables) based on Newton's method and a numerical solution of the Blasius equation. Wang [18] presented a new algorithm for solving the classical Blasius equation.

Hashim [11] studied the Adomian decomposition method for solving BVPs for fourth-order integro-differential equations showing that with a few modifications the Adomian's method can be used to obtain the known results of the special functions of mathematical physics and the Blasius equation [13].

Kechil et al. [15] applied a non perturbative solution of free-convective boundary-layer equation by ADM. Chang [9] presented a decomposition solution for fins with temperature dependent surface heat flux. Chiu and Chen [10] used a decomposition method for solving the convective longitudinal fins with variable thermal conductivity. ADM also have used by several researchers to solve a wide range of physical problems in various engineering fields such as fluid flow and porous media simulation [6, 12, 14, 17].

2. The Principle of the Adomian Decomposition Method (ADM) [3]

Beginning with an equation

$$Fu(\mathrm{t}) = g(\mathrm{t}) \tag{1}$$

where F represents a general nonlinear ordinary differential operator involving both linear and nonlinear terms. The linear term is decomposed into $L + R$, where L is easily invertible and R is the remainder of the linear operator. For convenience, L may be taken as the highest order derivative which avoids difficult integrations which result when complicated Green's functions are involved. Thus the equation (1) can be written as

$$Lu+Ru+Nu=g \tag{2}$$

where Nu represents the nonlinear terms. Solving for Lu,

$$Lu=g–Ru–Nu. \tag{3}$$

Because L is invertible, operating with its inverse L^{-1} yields

$$L^{-1}Lu= L^{-1}g– L^{-1}Ru– L^{-1}Nu \tag{4}$$

An equivalent expression is

$$u=\Phi+ L^{-1}g– L^{-1}Ru– L^{-1}Nu \tag{5}$$

where Φ is the integration constant and satisfies $L\Phi$=0. For initial-value problems we conveniently L^{-1} for $L = \frac{d^n}{dt^n}$ as the n-fold definite integration operator from t_0 to t. For the operator $L = \frac{d^n}{dt^n}$ for example, we have $L^{-1}Lu = u - u(0) - tu'(0)$ and therefore

$$u = u(0) + tu'(0) + L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{6}$$

For boundary value problems, indefinite integrations are used and the constants are evaluated from the given conditions. Solving for u yields

$$u = A + Bt + L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{7}$$

The Adomian decomposition method [22] assumes an infinite series solution for unknown function u given by

$$u = \sum_{n=0}^{\infty} u_n \tag{8}$$

and the nonlinear term *Nu*, assumed to be analytic function *f*(*u*), is decomposed as follows:

$$Nu = f(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, ..., u_n) \tag{9}$$

where A_n are the appropriate Adomian's polynomials. These A_n polynomials depend on the particular nonlinearity and these A_n Adomian polynomials are calculated by the general formula

$$A_n(u_0, u_1, ..., u_n) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]\right]_{\lambda=0} \quad , n \geq 0 \tag{10}$$

Substituting eq. (8) and eq. (9) into eq. (5) gives

$$\sum_{n=0}^{\infty} u_n = \Phi + L^{-1}g - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n \tag{11}$$

Each term of series (8) is given by the recurrence relation

$$\begin{aligned}
u_0 &= \Phi + L^{-1}g \\
u_1 &= -L^{-1}Ru_0 - L^{-1}A_0 \\
u_2 &= -L^{-1}Ru_1 - L^{-1}A_1 \\
u_3 &= -L^{-1}Ru_2 - L^{-1}A_2 \\
&\vdots \\
u_n &= -L^{-1}Ru_{n-1} - L^{-1}A_{n-1}
\end{aligned} \tag{12}$$

where *An* are the special Adomian polynomials or equivalently

$$\begin{aligned}
A_0 &= f(u_0), \\
A_1 &= u_1 f'(u_0), \\
A_2 &= u_2 f'(u_0) + \frac{1}{2}u_1^2 f''(u_0) \\
A_3 &= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{1}{3!}u_1^3 f''''(u_0) \\
A_4 &= u_4 f'(u_0) + \left(u_1 u_3 + \frac{1}{2}u_2^2\right) f''(u_0) + \frac{1}{2}u_1^2 u_2 f''''(u_0) + \frac{1}{24}u_1^4 f^{(iv)}(u_0) \\
&\vdots
\end{aligned} \tag{13}$$

So, the practical solution for the *n*-term approximation is,

$$\varphi_n = \sum_{i=0}^{n-1} u_i \quad , n \geq 1 \tag{14}$$

and the exact solution is

$$u = \lim_{n\to\infty} \varphi_n = \sum_{i=0}^{\infty} u_i \tag{15}$$

3. The Adomian decomposition method applied to Newell-Whitehead model

The nonlinear wave equation with dissipation and nonlinear transport term is given as: [15], [16]

$$\alpha u u_x + \beta u_{tt} + \gamma u_t = u_{xx} - f(u) \;\; ; f(u) = (u - a_1)(u - a_2)(u - a_3) \tag{16}$$

Where a_1, a_2, a_3 are distinct real numbers and α, β, γ , are constants.
For $\alpha = 0, \beta = 0, \gamma = 1$, Eq. (16) reduces to the nonlinear reaction- diffusion form

$$u_t = u_{xx} - f(u) \;\; ; f(u) = (u - a_1)(u - a_2)(u - a_3) \tag{17}$$

for different choices of the parameters a1, a2, a3 Eq. (17) reduces to the well known nonlinear reaction diffusion equations appearing in many different branches of sciences: when $a_1 = 0, a_2 = 1, a_3 = -1$ we get the Newell-Whitehead equation [20]

$$\frac{\partial u}{\partial t} = \Delta u + u(1 - u^2) \qquad (t, x) \in (0, \infty) \times \Omega \tag{18}$$

with the initial and boundary conditions

$$u(x,0) = u_0(x), \quad x \in \Omega$$

$$u(x,t) = 0, \quad t > 0 \;,\; x = 0 \;\; and \quad x = L \tag{18a}$$

$$\frac{\partial u}{\partial x} = 0 \quad at \quad x = 0 \;\; and \quad x = L.$$

which arises after carrying out a suitable normalization in the study of thermal convection of a fluid heated from below. Considering the Perturbation from a stationary state, the equation describes the evolution of the amplitude of the vertical velocity if this is a slowly varying function of time *t* and position *x*.

The equation (18) written in an operator form

$$L_t u = u_{xx} + u - N(u) \tag{19}$$

where $L_t = \frac{\partial}{\partial t}$ and $N(u) = u^3$ is nonlinear term. Applying the inverse operator to the equation (19) and using the initial data (18a) yields

$$u(x,t) = u_0(x) + L^{-1} u_{xx} + L^{-1} u - L^{-1} N(u) \tag{20}$$

The ADM suggests the solution $u(x,t)$ be decomposed by infinite series of components

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{21}$$

and the nonlinear operator $N(u)$ by the infinite series of the Adomian polynomials

$$N(u) = \sum_{n=0}^{\infty} A_n \tag{22}$$

The first four components of Adomain polynomials according to (13) read

$$A_0 = u_0^3$$

$$A_1 = 3u_0^2 u_1$$
$$A_2 = 3u_0^2 u_2 + 3u_0 u_1^2$$
$$A_3 = 3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3 \tag{23}$$

From eq. (21) and eq. (22), the iterates are determined by the following recursive way:

$$\begin{aligned} u_0 &= u_0(x) \\ u_n &= L^{-1}(\Delta u_{n-1} + u_{n-1}) - L^{-1}(A_{n-1}), \ \ n \geq 1. \end{aligned} \tag{24}$$

The decomposition method provides a reliable technique that requires less work if compared with traditional techniques.

4. Application and Numerical Results

To give a clear overview of the methodology, the following example will be discussed. All the results are calculated by using the MATLAB 7.4 software.

Consider the Newell-Whitehead equation [11]

$$\frac{\partial u}{\partial t} = \Delta u + u(1 - u^2)$$

with the initial conditions

$$u(x,0) = -\sqrt{\frac{a}{b}}\ \frac{c_1 e^{(0..5(\sqrt{2a})x)} - c_2 e^{(-0.5(\sqrt{2a})x)}}{(c_1 e^{(0.5(\sqrt{2a})x)} + c_2 e^{(-0.5(\sqrt{2a})x)} + 1)} \tag{25}$$

And exact Solution

$$u(x,t) = -\sqrt{\frac{a}{b}}\ \frac{c_1 e^{(0..5(\sqrt{2a})x)} - c_2 e^{(-0.5(\sqrt{2a})x)}}{(c_1 e^{(0.5(\sqrt{2a})x)} + c_2 e^{(-0.5(\sqrt{2a})x)} + c_3 e^{(-3/2)at})}$$

Substitution the initial condition eq. (25) into eq. (24) and using eq. (23) to calculate the Adomian polynomials, yields the following recursive relation

$$u_0 = -\sqrt{\frac{a}{b}}\ \frac{c_1 e^{(0..5(\sqrt{2a})x)} - c_2 e^{(-0.5(\sqrt{2a})x)}}{(c_1 e^{(0.5(\sqrt{2a})x)} + c_2 e^{(-0.5(\sqrt{2a})x)} + 1)}$$

$$u_n = L^{-1}(\Delta u_{n-1} + u_{n-1}) - L^{-1}(A_{n-1}), \ \ n \geq 1 \tag{26}$$

We will take $a = b = C_1 = C_2 = C_3 = 1$. The first few terms of the decomposition series are given by:

$$u_0 = (-e^{\left(\frac{1}{2}\sqrt{2}\,x\right)} + e^{\left(-\frac{1}{2}\sqrt{2}\,x\right)})/((e^{\left(\frac{1}{2}\sqrt{2}\,x\right)} + e^{\left(-\frac{1}{2}\sqrt{2}\,x\right)}) + 1)$$

$$u_1 = 3/2(\left(-e^{\left(\frac{1}{2}\sqrt{2}\,x\right)} + e^{\left(-\frac{1}{2}\sqrt{2}\,x\right)}\right) t)/(e^{(\frac{1}{2}\sqrt{2}\,x)} + e^{(-\frac{1}{2}\sqrt{2}\,x)} + 1)^2$$

$$u_2 = \frac{-9/8(t^2\left(-e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}+e^{\left(-\frac{1}{2}\sqrt{2}\,x\right)}\right)\left(-e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}+e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}\right)-1)}{\left(e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}+e^{\left(-\frac{1}{2}\sqrt{2}\,x\right)}+1\right)^3}$$

$$u_3 = \frac{9/16(t^3\left(-e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}+e^{\left(-\frac{1}{2}\sqrt{2}\,x\right)}\right)\left(-e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}-3+4e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}-e^{(\sqrt{2}x)}+4e^{(-0.5\sqrt{2}x)}\right))}{\left(e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}+e^{\left(-\frac{1}{2}\sqrt{2}\,x\right)}+1\right)^4} \tag{27}$$

other components are determined similarly (we will use four).
Substituting relations (27) into recursive relation (21) yields

$$u = \frac{-e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}+e^{\left(-\frac{1}{2}\sqrt{2}\,x\right)}}{\left(e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}+e^{\left(-\frac{1}{2}\sqrt{2}\,x\right)}\right)+1} + 3/2\left(\left(-e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}+e^{\left(-\frac{1}{2}\sqrt{2}\,x\right)}\right)t\right)/\left(e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}+e^{\left(-\frac{1}{2}\sqrt{2}\,x\right)}+1\right)^2$$

$$\frac{-9/8\left(t^2\left(-e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}+e^{\left(-\frac{1}{2}\sqrt{2}\,x\right)}\right)\left(-e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}+e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}\right)-1\right)}{\left(e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}+e^{\left(-\frac{1}{2}\sqrt{2}\,x\right)}+1\right)^3}+$$

$$\frac{9/16\left(t^3\left(-e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}+e^{\left(-\frac{1}{2}\sqrt{2}\,x\right)}\right)\left(-e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}-3+4e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}-e^{(\sqrt{2}x)}+4e^{(-0.5\sqrt{2}x)}\right)\right)}{\left(e^{\left(\frac{1}{2}\sqrt{2}\,x\right)}+e^{\left(-\frac{1}{2}\sqrt{2}\,x\right)}+1\right)^4} \quad (28)$$

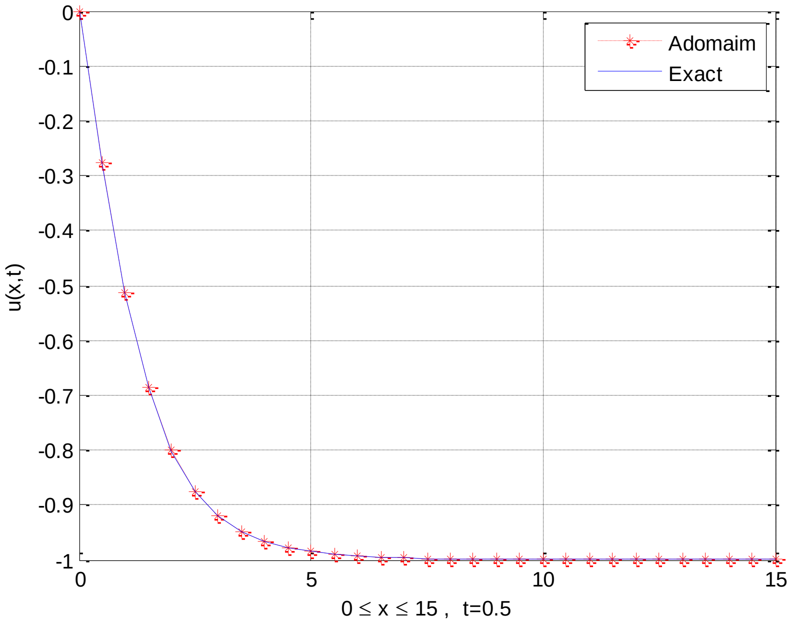

Fig.1: Comparison of exact and Adomain solution for t=0.5

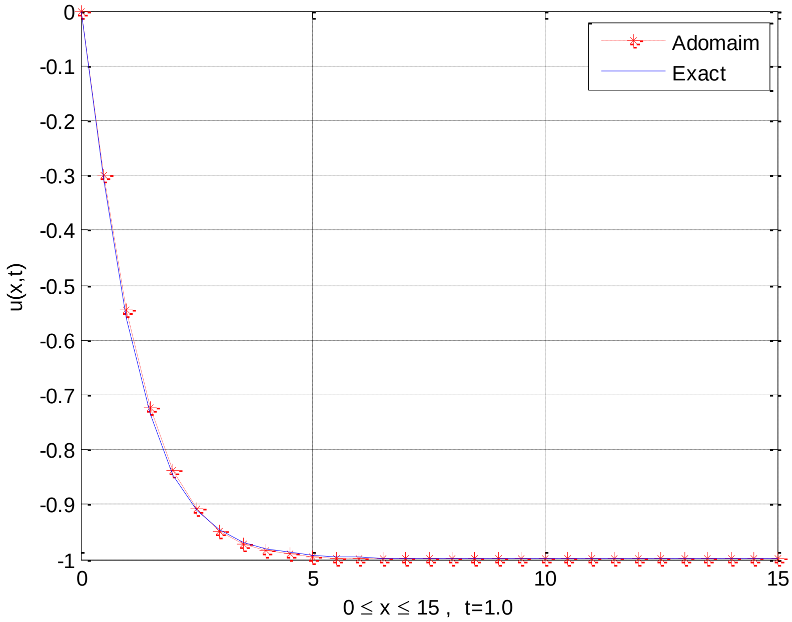

Fig.2: Comparison of exact and Adomain solution for t=1.0

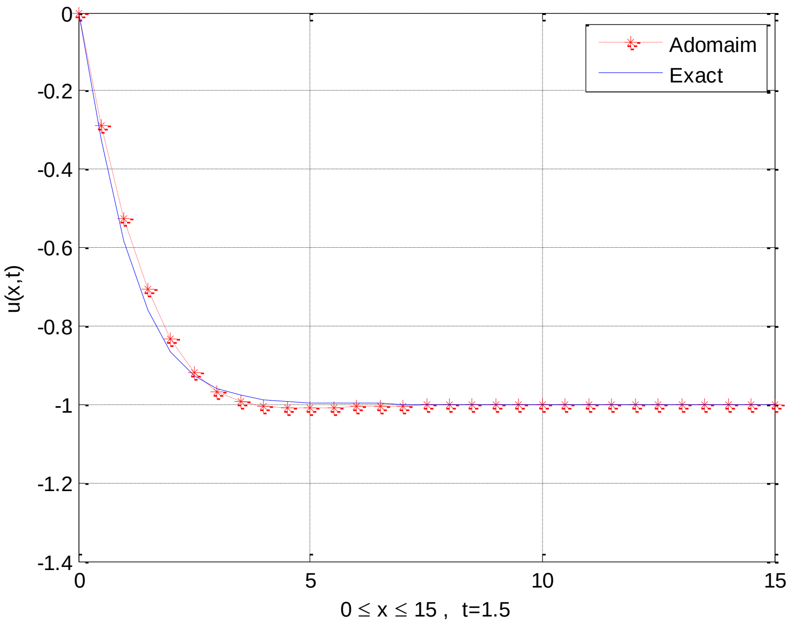

Fig.3: Comparison of exact and Adomain solution for t=1.5

Table 1: Comparison of the exact and Adomain decomposition solutions

	Adomain Decomposition u(x,t)	Exact Solution u(x,t)	Absolute Error	Adomain Decomposition u(x,t)	Exact Solution u(x,t)	Absolute Error	Adomain Decomposition u(x,t)	Exact Solution u(x,t)	Absolute Error
	$0 \le x \le 15$ and t=0.5		$10^{-3}\times$	$0 \le x \le 15$ and t=1.0			$0 \le x \le 15$ and t=1.5		
X=0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
X=0.5	-0.2772	-0.2778	0.5581	-0.2991	-0.3073	0.0082	-0.2869	-0.3235	0.0365
X=1.0	-0.5118	-0.5128	0.9418	-0.5462	-0.5594	0.0132	-0.5274	-0.5844	0.0570
X=1.5	-0.6848	-0.6858	0.9834	-0.7222	-0.7352	0.0130	-0.7078	-0.7611	0.0533
X=2.0	-0.8008	-0.8015	0.7105	-0.8365	-0.8451	0.0086	-0.8349	-0.8674	0.0325
X=2.5	-0.8745	-0.8748	0.3278	-0.9065	-0.9097	0.0032	-0.9182	-0.9272	0.0089
X=3.0	-0.9203	-0.9203	0.0144	-0.9476	-0.9467	0.0009	-0.9678	-0.9597	0.0080
X=3.5	-0.9487	-0.9485	0.1685	-0.9710	-0.9679	0.0031	-0.9939	-0.9773	0.0166
X=4.0	-0.9664	-0.9662	0.2401	-0.9840	-0.9802	0.0038	-1.0056	-0.9869	0.0187
X=4.5	-0.9777	-0.9774	0.2428	-0.9911	-0.9874	0.0037	-1.0096	-0.9922	0.0174
X=5.0	-0.9850	-0.9848	0.2127	-0.9950	-0.9919	0.0031	-1.0098	-0.9952	0.0145
X=5.5	-0.9898	-0.9896	0.1723	-0.9971	-0.9946	0.0025	-1.0085	-0.9970	0.0115
X=6.0	-0.9930	-0.9928	0.1330	-0.9983	-0.9964	0.0019	-1.0068	-0.9981	0.0087
X=6.5	-0.9952	-0.9951	0.0997	-0.9990	-0.9976	0.0014	-1.0052	-0.9987	0.0065
X=7.0	-0.9966	-0.9966	0.0732	-0.9994	-0.9983	0.0010	-1.0039	-0.9992	0.0047
X=7.5	-0.9977	-0.9976	0.0530	-0.9996	-0.9988	0.0007	-1.0028	-0.9994	0.0034
X=8.0	-0.9984	-0.9983	0.0380	-0.9997	-0.9992	0.0005	-1.0020	-0.9996	0.0024
X=8.5	-0.9989	-0.9988	0.0271	-0.9998	-0.9994	0.0004	-1.0015	-0.9997	0.0017
X=9.0	-0.9992	-0.9992	0.0192	-0.9999	-0.9996	0.0003	-1.0010	-0.9998	0.0012
X=9.5	-0.9994	-0.9994	0.0136	-0.9999	-0.9997	0.0002	-1.0007	-0.9999	0.0009
X=10.0	-0.9996	-0.9996	0.0096	-0.9999	-0.9998	0.0001	-1.0005	-0.9999	0.0006
X=10.5	-0.9997	-0.9997	0.0068	-1.0000	-0.9999	0.0001	-1.0004	-0.9999	0.0004
X=11.0	-0.9998	-0.9998	0.0048	-1.0000	-0.9999	0.0001	-1.0003	-1.0000	0.0003
X=11.5	-0.9999	-0.9999	0.0033	-1.0000	-0.9999	0.0000	-1.0002	-1.0000	0.0002
X=12.0	-0.9999	-0.9999	0.0024	-1.0000	-1.0000	0.0000	-1.0001	-1.0000	0.0001
X=12.5	-0.9999	-0.9999	0.0017	-1.0000	-1.0000	0.0000	-1.0001	-1.0000	0.0001
X=13.0	-1.0000	-1.0000	0.0012	-1.0000	-1.0000	0.0000	-1.0001	-1.0000	0.0001
X=13.5	-1.0000	-1.0000	0.0008	-1.0000	-1.0000	0.0000	-1.0000	-1.0000	0.0001
X=14.0	-1.0000	-1.0000	0.0006	-1.0000	-1.0000	0.0000	-1.0000	-1.0000	0.0000
X=14.5	-1.0000	-1.0000	0.0004	-1.0000	-1.0000	0.0000	-1.0000	-1.0000	0.0000
X=15.0	-1.0000	-1.0000	0.0003	-1.0000	-1.0000	0.0000	-1.0000	-1.0000	0.0000

It is clear from the figures (1-3) and the table that Adomain decomposition method is so accurate and converging to the exact solution.

5. Conclusions

The Adomain decomposition method is effective and powerful method for solving nonlinear partial differential Newell-Whitehead equations. The important part of this method is calculating Adomain polynomials for nonlinear operator.

***Acknowledgements**:* Words would not be enough to express my deep feeling of gratitude and immense indebtedness to my wife for her continuous support and encouragement on this work.

REFERENCES

[1] Abbasbandy S., "A numerical solution of Blasius equation by Adomian's decomposition method and comparison with homotopy perturbation method", Chaos Solitons Fract 2007; 31:257–60.

[2] Abbasbandy S., "Extended Newton's method for a system of non-linear equations by modified Adomian decomposition method", Appl Math Comput 2005; 170:648–56.

[3] Adomian G., "An analytical solution of the stochastic Navier–Stokes system", Found Phys 1991; 21(7):831–43.

[4] Adomian G., "Solving frontier problems of physics: the decomposition method", Boston: Kluwer Academic Publishers; 1994.

[5] Adomian G., "Nonlinear Stochastic Operator Equations", Academic Press, Orlando, 1986.

[6] Adomian G., "A review of the decomposition method in applied mathematics", J Math Anal. Appl 1988;135:501–44.

[7] Adomian G., "Stochastic Systems", Academic Press, New York, 1983.

[8] Allan FM, Syam MI., "On the analytic solutions of the non homogeneous Blasius problem", J. Comput Appl Math. 2005.

[9] Chang MH., "A decomposition solution for fins with temperature dependent surface heat flux", Int J Heat Mass Transfer 2005; 1819:24-48.

[10] Chiu CH, Chen CK., "A decomposition method for solving the convective longitudinal fins with variable thermal conductivity", Int J Heat Mass Transfer 2002; 45:2067–75.

[11] Cicogna, G., "Weak" symmetries and adopted variables for differential equations", *Int. J. Geometric Meth. Modern Phys.,* Vol. 1, No 1–2, pp. 23–31, 2004.

[12] Hashim I., "Adomian decomposition method for solving BVPs for fourth-order integro- differential equations", J Comput Appl Math 2006; 658:64-193.

[13] Hashim I., "Comments on A new algorithm for solving classical Blasius equation", J Comput Appl Math 2005; 182:362–71.

[14] Kaya D, Yokus A., "A decomposition method for finding solitary and periodic solutions for a coupled higher-dimensional Burgers equations", Appl Math Comput 2005; 164:857–64.

[15] Kechil, S.A. and Hashim I., "Non-perturbative solution of free-convective boundary-layer equation by Adomian decomposition method ", Phys Lett A 2007; 363:110.

[16] Newell A.C. and Whitehead J.A. Whitehead, "*Finite bandwidth, _nite amplitude convection"*, J. Fluid Mech. 38 (1969), 279-303.

[17] Pamuk S., "Solution of the porous media equation by Adomian's decomposition method", Phys Lett A 2005; 344:184–8.

[18] Wang L., "A new algorithm for solving classical Blasius equation", Appl Math Comput; 2004; 157:1–9.

[19] Wazwaz AM., "Approximate solutions to boundary value problems of higher order by the modified decomposition method", Comput Math Appl 2000; 40(6–7): 679–91.

[20] Wazwaz AM., "The decomposition method applied to systems of partial differential equations and to the reaction–diffusion Brusselator model", Appl Math Comput 2000; 110(2–3):251–64.

[21] Wazwaz AM., "The modified decomposition method applied to unsteady flow of gas through a porous medium", Appl. Math. Comput. 2001, 118(2-3): 123-32.

[22] Wazwaz AM., "The numerical solution of sixth-order boundary value problems by the modified decomposition method", Appl Math Comput 2001; 118:311–25.