# On n-Weakly Regular Rings

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## ABSTRACT

As a generalization of right weakly regular rings, we introduce the notion of right n-weakly regular rings, i.e. for all a$\in$N(R), a$\in$aRaR. In this paper, first give various properties of right n-weakly regular rings. Also, we study the relation between such rings and reduced rings by adding some types of rings, such as NCI, MC2 and SNF rings.

Keywords: Weakly Regular Rings , Reduced Rings.

**حول حلقات المنتظمة بضعف** n–

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**الملخص**

كتعميم للحلقات المنتظمة بضعف، نُعَرِّف الحلقات المنتظمة بضعف من النمط n على إنها لكل $a \in N(R)$ فأن $a \in aRaR$ وتسمى هذه الحلقة حلقة منتظمة بضعف من النمط n. في هذا البحث أعطينا خواص متنوعة للحلقات المنتظمة بضعف من النمط n وكذلك درسنا العلاقة بين تلك الحلقات والحلقات المختزلة بإضافة بعض أنواع الحلقات ومنها الحلقات من النمط NCI وMC2 وSNF .

الكلمات المفتاحية : الحلقات المنتظمة بضعف، الحلقات المختزلة.

## .Introduction:

Throughout this paper a ring *R* denotes an associative ring with identity and all modules are unitary. For a subset *X* of R, the left(right) annihilator of *X* in *R* is denoted by *r(X)(l(X))*. If *X={a}*, we usually abbreviate it to *r(a)(l(a))*. We write *J(R), and N(R),* for the Jacobson radical and the set of nilpotent elements respectively.

The center of the ring R is denoted by *Cent(R)* and it is $Cent(R) = \{a \in R / ar = ra \quad \forall \quad r \in R\}$. A ring R is called n-regular if for all a$\in N(R)$, a$\in$aRa [7]. A right *R*-module *M* is called *N flat* if for any $a \in N(R)$, the mapping $1_M \otimes i: M \otimes_R Ra \to M \otimes_R R$ is monic, where $i : Ra \to R$ is the inclusion mapping [8]. A ring R is called right (left) SNF if every simple right(left) R-module is N flat [8].

A ring R is called *semiprime* if it has no nilpotent ideals [6]. A ring R is called *reduced* if *N(R)* = 0 [6], or equivalently, $a^2=0$ implies $a=0$ in R for all $a \in R$. Recall that a ring R is *MERT*(resp. *MELT*), if every maximal essential right (resp. left) ideal of R is an ideal [9].

**2. n -Weakly Regular Ring**

This section is devoted to give the definition of n-weakly regular rings with some of its characterizations and basic properties.

A ring R is *right (left) weakly regular* [6], if $a \in aRaR$ (RaRa) for every $a \in R$. We called R is weakly regular if it is both right and left weakly regular.

**Definition 2.1**

A ring R is to be *right* (left) *n-weakly regular* if $a \in aRaR$ ($a \in RaRa$) for all $a \in N(R)$. We say that R is n-weakly regular if it is right and left n-weakly regular ring.

Clearly every weakly regular rings are *n-weakly regular*.

**Examples:**

1- Every reduced ring is n-weakly regular.
2- Every n-regular ring is n-weakly regular ring.
3- The ring $Z_6$ of integers modulo 6, is reduced, n-regular, weakly regular ring, so it is n- weakly regular.
4- Let $R=\left\{\begin{bmatrix} Z_2 & Z_2 \\ Z_2 & Z_2 \end{bmatrix}\right\}$, $N(R)=\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$. *R is* n-regular, weakly regular ring, so it is n- weakly regular but R not reduced ring.
5- The ring *Z* of integer number is reduced, n-regular, so it is n- weakly regular but Z is not weakly regular ring.

**Proposition 2.2**

R is a right n-weakly regular ring if and only if aR is idempotent right ideal for all $a \in N(R)$.

Proof:

Let R is a right n-weakly regular ring and I is a principal right ideal of R generated by a nilpotent element, then there exists $a \in N(R)$ such that I=aR, clearly $I^2 \subseteq I$.

On the other hand, since R is right n-weakly, then there exists $y,z \in R$ such that a=ayaz. Now let $x \in I$, then there exists $r \in R$ such that $x=ar=ayazr \in I^2$. Therefore $I \subseteq I^2$. Hence $I^2 = I$.

Conversly, Let $a \in N(R)$, since aR is idempotent right ideal of R, so $a \in aR=aRaR$. Therefore R is right n-weakly regular ring. ■

**Proposition 2.3**

Let R be a right n-weakly regular ring. If $aR \subseteq I$, for $a \in N(R)$ and I is right or left ideal. Then aRI=aR.

Proof:

It is clearly that $bRI \subseteq bR$ for any $b \in R$. Now let $a \in N(R)$ and $x \in aR$, then there exists $r \in R$ such that x=ar. Since R is right n-weakly regular ring then there exists $y,z \in R$ such that a=ayaz, x=ayazr, hence $azr \in aR \subseteq I$. So $aR \subseteq aRI$. Therefore aRI=aR. ■

**Corollary 2.4**

Let R be a ring for all $a \in N(R)$ and any I right or left ideal of R such that $aR \subseteq I$. Then the following condition are equivalent:

1- R is right n-weakly regular.
2- For all $a \in N(R)$, aRI=aR.

**Proof:**

$1 \to 2$ by Proposition 2.3.
$2 \to 1$ Let I=aR and by Proposition 2.2. ■

**Proposition 2.5**

Let *R* be a right n-weakly regular ring. Then $N(R) \cap Cent(R)=0$

**Proof :**

If $N(R) \cap Cent(R) \neq 0$, then there exists $0 \neq a \in N(R) \cap Cent(R)$ such that $a^2=0$. since R is right n-weakly regular then there exists $y,z \in R$ such that $a=ayaz=a^2yz=0yz=0$ ($a \in Cent(R)$). Therefore *a*=0. This is shows that $N(R) \cap Cent(R)=0$. ■

**Corollary 2.6**

Let R be a commutative ring. Then R is reduced if and only if R is right n-weakly regular.

**Lemma 2.7 [2]**

1- Every one sided or two sided nil ideal of R is contained in J(R).
2- In any ring R, $a \in J(R)$ if and only if 1-ar is invertible for all $r \in R$.

Now we have the following proposition

**Theorem 2.8**

Let R be a right n-weakly regular ring. Then $N(R) \cap J(R)=0$.

**Proof:**

If $a \in N(R) \cap J(R)$, then there exists $y,z \in R$ such that a= ayaz. Hence a(1-yaz)=0, since $a \in J$, yaz $\in J$, then by Lemma 2.7(2), there exists invertible element $v \in R$ such that (1-yaz)v=1. So (a-ayaz)v=a, yield a=0. Therefore $N(R) \cap J(R)=0$. ■

Let R be a ring we denoted to the upper nil radical for a ring R by $Nil^*(R)$ and it is the sum of nil ideal in the ring R.

**Corollary 2.9**

Let R be a right n-weakly regular ring .Then $Nil^*(R)=0$.

**Proof:**

Let I be a left or right or two sided nil ideal, by Lemma 2.7(1), we have that $I \subseteq J(R)$, $I \subseteq N(R) \cap J(R)=0$, Theorem 2.8, I=0, which is a contradiction. So R not contain any nil ideal. Therefore $N^*(R)=0$. ■

**Corollary 2.10**

Let R be a right n-weakly regular ring. Then R is semiprime ring.

**Proof:**

Let I be a nilpotent right ideal then $I \subseteq N(R) \cap J(R)=0$ (Theorem 2.8*)* I=0. Therefore R is semiprime ring. ■

## 3. The Connection between n-Weakly Regular Rings and Other Rings.

In this section we gives the connection between n-weakly regular rings and reduced rings, SNF rings.

**Proposition 3.1**

The following conditions are equivalent for a ring R.

1- R is reduced.
2- R is right n-weakly regular and N(R) forms a right ideal of R.
3- R is right n-weakly regular and N(R) forms a left ideal of R.
4- R is right n-weakly regular and NI ring.
5- R is right n-weakly regular and N(R) $\subseteq$J(R).

**Proof:**

$1\rightarrow 4\rightarrow 3\rightarrow 5$, $1\rightarrow 2\rightarrow 5$ it is trivial.

$5\rightarrow 1$

Suppose that R is right n-weakly regular ring. So N(R) $\cap$J(R)=0, (Theorem 2.8). Since N(R) $\subseteq$J(R), then N(R) $\cap$J(R)= N(R) =0. Therefore R is reduced. ■

**Theorem 3.2**

Let R be a ring with aR=Ra, for all a$\in$N(R). Then the following conditions are equivalent:

1- R is right n-weakly regular.
2- R is n-regular.
3- R is reduced.

**Proof:**

$1\rightarrow 2$

Let $0\neq a\in R$ , such that $a^2$=0. Since R is a right n-weakly regular, then a$\in$aR=aRaR (Proposition 2.1)

= aRRa (Ra=aR)

=aRa.

so a$\in$aRa, hence R is n-regular.

$2\rightarrow 3$

Let $0\neq a\in R$ , such that $a^2$=0. Since R is a right n-regular, then there exists b$\in$R such that a=aba since aR=Ra there exists x$\in$R such that ab=xa, so $a=aba=xa^2=x0=0$. Therefore R is reduced.

$3\rightarrow 1$ It is trivial. ■

A ring R is called NCI provided that N(R) contains a non zero ideal of R whenever N(R)$\neq$0 [1].

**Lemma 3.3 [1]**

Let R be a ring with N(R)$\neq$0. Then R is NCI if and only if N*(R) $\neq$0.

**Proportion 3.4**

Let R be a NCI ring , then R is right n-weakly regular if and only if R is reduced.

**Proof:**

Let N(R) $\neq$0, sinc R is NCI ring from Lemma 3.3, we get that N*(R) $\neq$0 but R is right n-weakly regular, N*(R)=0 (Corollary 2.9), which is contradiction. So N(R)=0. Therefore R is reduced. ■

A ring R is called weakly reversible if and only if for all a,b,r$\in$R such that ab=0, Rbra is a nil left ideal of R (equivalently braR is nil right ideal of R). Clearly ZI ring is weakly reversible [4].

**Proposition 3.5**

A ring R be right n-weakly regular ring and weakly reversible if and only if R is reduced.

**Proof:**

Let $a \in R$ with $a^2=0$. Then a=ayaz for some $y,z \in R$ because R is right n-weakly regular ring. Since R is weakly reversible then ayaR is nil right ideal of R so ayaR$\subseteq J(R) \cap N(R) = 0$ (Lemma 2.7(1) & Theorem 2.8) we get ayaR=0, in particular a=ayaz=0. Therefore R is reduced.

Converse, it is trivial. ■

Recall that a ring R is right *MC2* if $K \cong eR$ is simple, $e^2=e$, , then K=gR for some $g^2=g$ [5].

**Lemma 3.6 [9]**

Let R be a left MC2,right SNF ring and MELT ring. Then R is a semiprime ring and right non singular.

**Theorem 3.7 [9]**

Let I be a right ideal of a ring R. Then R/I is N flat if and only if Ia=I$\cap$Ma for all $a \in N(R)$.

**Theorem 3.8**

Let R be right SNF, left MC2 and MELT ring. Then R is left n-weakly regular ring.

**Proof:**

From Lemma 3.6, we get that R is a semiprime ring and $a \in N(R)$. If RaR+l(a) $\neq$R, then there exists a maximal left ideal M of R containing RaR+l(a), if M is not essential then we can write M=l(e), where $e^2=e \in R$ and $e \neq 0$, since RaRe=0 because RaR$\subseteq$M, $(ReRa)^2=0$ implies ReRa=0 (since R is semiprime) ReRa=0 in particular ea=0 and $e \in$ l(a) $\subseteq$M=l(e), then $e^2=0$, which is a contradiction. Therefore M is an essential, since R is MELT ring, then M is a two sided ideal then there exists a maximal right ideal L in R containing M, since R is right SNF ring then R/ L is N flat right R-module, a=ma for some $m \in M$ (Theorem 3.7), (1-m)a=0, 1-m$\in$l(a) $\subseteq M \subseteq L$ therefore 1-m$\in$ L implies $1 \in$ L which is a contradiction, therefore RaR+l(a) =R for all $a \in N(R)$. Thus R is left n-weakly regular ring. ■

**Theorem 3.9**

Let R be MELT and right SNF ring with RaR is essential for all $a \in N(R)$, Then R is left n-weakly regular ring.

**Proof:**

Let $a \in N(R)$. If RaR+l(a) $\neq$R then there exists a maximal left ideal M of R containing RaR+l(a), since RaR is left annihilator of a nilpotent element by the hypothesis RaR is essential left ideal in R, M is an essential ideal of R (MELT ring), there exists a maximal right ideal L in R such that $M \subseteq L$. Since R is right SNF ring, then R/ L is N flat right R-module, a=ma for some $m \in M$ (Theorem 3.7), 1-m$\in$l(a) $\subseteq M \subseteq L$ implies $1 \in$ L, which is a contradiction. Therefore RaR+l(a) =R for all $a \in N(R)$, and so R is left n-weakly regular ring. ■

**Definition 3.10 [8]**

A right *R*-module *M* is said to be *nil*-injective, if for any $a \in N(R)$, any right *R*-homomorphism $f:aR \to M$ can be extended to $R \to M$, or equivalently $f=m.$, where $m \in M$.

The ring R is called right nil-injective if $R_R$ is right nil-injective. Clearly a reduced ring is a right nil-injective and n-regular ring is a right nil-injective [8].

**Theorem 3.11**

Let R be a semiprime ring whose simple singular right R-module are nil-injective. Then R is right n-weakly regular ring.

**Proof:**

Let $a \in N(R)$. We claim that RaR+r(a)=R if not, there exists a maximal right ideal M of R containing RaR+r(a). If M is not essential in R then M=r(e), $e^2=e \in R$. Since $Rae \subseteq RaR \subseteq M=r(e)$, eRae=0, $(aeR)^2=0$ but R is semiprime, aeR=0, so ae=0. Thus $e \in r(a) \subseteq M=r(e)$, which is a contradiction. Hence M is essential right ideal in R and so R/M is nil-injective. Define a mapping $f : aR \to R/M$ such that f(ar)=r+M, let $x,y \in R$ such that ax=ay, a(x-y)=0, (x-y)+M=f(a(x-y))=f(0)=M, then $x-y \in M$, x+M=y+M, f(ax)=x+M=y+M=f(ay), f is well define. 1+M=f(a)=(b+M)(a+M)=ba+M, $1-ba \in M$, since $ba \in RaR \subseteq M$ then $1 \in M$ which is a contradiction, so RaR+r(a)=R, in particular there is $y,z \in R$ and $v \in r(a)$ such that yaz+v=1, ayaz+av=a, a=ayaz. Therefore R is right n-weakly regular ring. ■

**Proposition 3.12**

Let R be a ring whose simple right R-module are nil-injective. Then R is right n-weakly regular ring.

**Proof:**

Assume that $a \in R$ such that aRa=0. Then $RaR \subseteq r(a)$. If $a \neq 0$ then there exists a maximal right ideal M containing r(a). By hypothesis R/M is nil-injective. We define a mapping $f : aR \to R/M$ such that f(ar)=r+M, f is well define similar to Theorem 3.11, so there exists $b \in R$ such that 1+M=f(a)=ba+M, $1-ba \in M$ because $ba \in RaR \subseteq M$ then $1 \in M$ which is a contradiction, so a=0. Therefore R is a semiprime ring, by Theorem 3.11 we get that R is right n-weakly regular ring. ■

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