On m-Regular Rings

Zubayda M. Ibraheem **Maha F. Khalaf**
zubaida_almulla@yahoo.com
College of Computer Sciences and Mathematics
University of Mosul

Received on: 13/02/2011 **Accepted on: 16/05/2011**

ABSTRACT

As a generalization of regular rings, we introduce the notion, of m-regular rings, that is for all $a \in R$, there is a fixed positive integer m such that a^m is a Von-Neumann regular element. Some characterization and basic properties of these rings will be given. Also, we study the relation-ship between them and Von-Neumann regular rings, π-regular rings, reduced rings, locally rings, uniform rings and 2-primal rings.

Keyword: m-regular rings,regular rings, reduced rings, locally rings, uniform rings -π and 2-primal rings.

حول الحلقات المنتظمة من النمط−m

زبيدة محمد أبراهيم
z.mohammed@uomosul.edu.iq
كلية علوم الحاسوب والرياضيات، جامعة الموصل

تاريخ استلام البحث: 2011/2/13 **تاريخ قبول البحث: 2011/5/16**

الملخص

كتعميم للحلقات المنتظمة, قدمنا الحلقات المنتظمة من النمط − m على أنها لكل $a \in R$ يوجد عدد صحيح موجب ثابت m بحيث أن a^m يكون منتظم. في هذا البحث درسنا المميزات والخواص الأساسية له وكذلك العلاقة بين الحلقات المنتظمة من النمط − m والحلقات المنتظمة من النمط − π, والحلقات المختزلة والحلقات المحلية والحلقات الموحدة.

كلمات المفتاحية: حلقات منتظمة من النمط−m , حلقات منتظمة, حلقات مختزلة, حلقات محلية, حلقات موحدة, حلقات موحدة من النمط− π

1- Introduction

Throughout in this paper, R denotes an associative ring with identity. For a subset X of R, the right (left) annihilator of X in R is denoted by r(X) (l(X)). If X={a}, we usually abbreviate it to r(a) (l(a)). We write J(R), Y(R), Z(R), N(R), P(R) for the Jacobson radical, right singular ideal, left singular ideal, the set of all nilpotent element of R and the prime radical of R respectively.

A right R- module M is called p- Injective, if for any principal right ideal aR of R and any right R- homomorphism of aR into M can be extended to one of R into M. The ring R is called right p- Injective if R_R is p- Injective [12]. An ideal I of a ring R is said to be essential if and only if I has a non-zero intersection with every non-zero ideal of R. A ring R is called $\pi -$regular, if for each a in R , there exist a positive integer n and an element b in R such that $a^n = a^n b a^n$ [7]. A ring R is called reduced if $a^2 = 0$ implies $a = 0$ for all a in R [10]. A ring R is said to be reversible if $ab = 0$ implies $ba = 0$ for all a in R [3]. Finally a ring R is said to be right (left) duo if every right (left) ideal is a two-sided ideal of R [2].

2- m - Regular ring

This section is devoted to give the definition of m-regular rings with some of its characterization and basic properties.
A ring R is said to be Von-Neumann regular (or just regular) if and only if for each a in R there exists b in R such that a=aba [8].

Definition 2.1 [5]:
Let R be a ring, if there is a fixed positive integer $m \neq 1$ such that for all elements a of R, a^m is regular $\left(a^m = a^m b a^m\right)$. Then we say that R is m-regular, and it is left (right) m-regular if $a^m = x a^{m+1} \left(a^m = a^{m+1} y\right)$ for some $x, y \in R$. The ring R is (left or right) m-regular if all its elements have this property.

Examples : Z_3, Z_4, Z_8, Z_9 are m-regular rings.

Note: clearly that when $m = 1$, then R is regular ring, but the converse is not true by the following example :

Example [5]: The endomorphism ring of $G = Q \oplus \prod_p Z(p)$ is (left, right) 2-regular but not regular.

Proposition 2.2:
If y is an element of a ring R such that $a^m - a^m y a^m$ is regular element for a fixed positive integer $m \neq 1$, then a is m-regular .

Proof :
Let $x = a^m - a^m y a^m$
Since x is regular, then $x = xux$ for some $u \in R$.
Hence $a^m = x + a^m y a^m$

$$\begin{aligned} &= \left(a^m - a^m y a^m\right) u \left(a^m - a^m y a^m\right) + a^m y a^m \\ &= a^m \left(1 - y a^m\right) u \left(1 - a^m y\right) a^m + a^m y a^m \\ &= a^m \left[\left(1 - y a^m\right) u \left(1 - a^m y\right)\right] a^m + a^m y a^m \\ &= a^m \left[\left(1 - y a^m\right) u \left(1 - a^m y\right) + y\right] a^m \end{aligned}$$

Therefore $a^m = a^m z a^m$,where $z = \left(1 - y a^m\right) u \left(1 - a^m y\right) + y$.■

Theorem 2.3:
A ring R is m-regular if and only if $a^m R$ is generated by idempotent for every $a \in R$ and for a fixed positive integer $m \neq 1$.

Proof:
Let $a \in R$. Choose an idempotent e in R and there exists a fixed positive integer $m \neq 1$, such that $a^m R = eR$. Take $e = a^m b$ for some $b \in R$, then $a^m = ec$ for some c in R, so $e a^m = a^m b a^m$ and $e a^m = e.ec = ec = a^m$. Therefore $a^m = e a^m = a^m b a^m$. Thus R is m-regular.

Conversely: It is clear.■

Theorem 2.4:

If R is m- regular ring without zero divisor element, then R is a division ring.

Proof :

Let $0 \neq a \in R$. Since R is m- regular ring, then there exists b in R such that $a^m = a^m b a^m$, then

$0 = a^m - a^m b a^m = a^m(1 - ba^m) = a(a^{m-1}(1 - ba^m))$. Since $a \neq 0$, then $a^{m-1}(1 - ba^m) = 0$. So

$a(a^{m-2}(1 - ba^m)) = 0$

$\vdots$

$a(1 - ba^m) = 0$

So $1 - ba^m = 0$

Thus $1 = ba^m$, implies that $1 = (ba^{m-1})a$.

Hence a is a left invertible. Now, since $1 = (ba^{m-1})a$. Then $a = a(ba^{m-1})a$. Hence $(1 - aba^{m-1}) \in l(a) = 0$. So $1 = a(ba^{m-1})$, implies that a is a right invertible. Therefore R is a division ring. ■

Theorem 2.5:

If P is a primary ideal of a ring R , and if R/p is m-regular, then P is maximal.

Proof :

Let $a \in R$, then $a + p \in R/p$.

Since R/p is m-regular ring, then there exists $b + p \in R/p$ such that

$$a^m + p = (a+p)^m (b+p)(a+p)^m$$
$$= a^m b a^m + p.$$

So $a^m - a^m b a^m \in p$, thus $a^m(1 - ba^m) \in p$.

Suppose that $a^m \notin p$, then $(1 - ba^m)^n \in p, n \in z^+$.

Now, $(1 - ba^m)^n = 1 - \left[\sum_{k=1}^{n} c_k^n (-1)^{k-1} b^m a^{m(k-1)}\right] a^m \in p$.

Let $z = \sum_{k=1}^{n} c_k^n (-1)^{k-1} b^m a^{m(k-1)}$.

Then $1 - za^m \in p$ and so $1 + p = (z + p)(a^m + p)$. Therefore $a^m + p$ has an inverse and hence R/p is a division ring . Therefore P is maximal. ■

Theorem 2.6:

Let R be a ring with $r(a^{m+1}) \subseteq r(a^m)$ for a fixed positive integer $m \neq 1$ .Then R is m-regular if $R/r(a)$ is m-regular.

Proof :

Suppose that $R/r(a)$ is m-regular ring, then for any $a + r(a) \in R/r(a)$, there exists $b + r(a) \in R/r(a)$ such that

$$(a + r(a))^m = (a + r(a))^m (b + r(a))(a + r(a))^m$$

$a^m + r(a) = a^m b a^m + r(a)$. So $a^m - a^m b a^m \in r(a)$.

Hence $a(a^m - a^m b a^m) = 0$

That is $a^{m+1}(1-ba^m)=0$

So $1-ba^m \in r(a^{m+1}) \subseteq r(a^m)$

Hence $a^m(1-ba^m)=0$

Thus $a^m = a^m ba^m$.

Therefore R is m-regular. ■

Recall that, a ring R is called bounded index of nilpotency [4] if there exists a positive integer n such that $a^n = 0$, for all nilpotent elements a in R.

As a result of Theorem 2.6 we obtain the following corollary:

Corollary 2.7:

A ring R is m-regular if and only if R is bounded index of nilpotency and R/r(a) is m-regular for all $a \in R$.

Theorem 2.8:

Let I be an ideal of *R*. If R/I is a right m-regular and I is a right n-regular. Then *R* is right mn-regular.

Proof :

Let $x \in R$, then $x + I \in R/I$.

Since R/I is right m-regular, then there exists $y + I \in R/I$ such that:

$(x+I)^m = (x+I)^{m+1}(y+1)$ which implies that $x^m + 1 = x^{m+1}y + I$ and hence $x^m - x^{m+1}y \in I$. Since I is right n-regular ideal, then there exists $z \in I$, such that $(x^m - x^{m+1}y)^n = (x^m - x^{m+1}y)^{n+1} z$, implies that

$$x^{mn} - x^{mn-1}x^{m+1}y + x^{mn-2}\frac{(x^{m+1}y)^2}{2!} - \cdots + (x^{m+1}y)^n =$$

$$\left[x^{mn+m} - x^{mn}x^{m+1}y + x^{mn+m-2}\frac{(x^{m+1}y)^2}{2!} - \cdots + (x^{m+1}y)^{n+1}\right]z$$

Then

$$x^{mn} = x^{mn-1}x^{m+1}y - x^{mn-2}\frac{x^{2m+2}y^2}{2!} + \cdots - x^{mn+n}y^n +$$

$$\left[x^{mn+n} - x^{mn}x^{m+1}y + x^{mn+m-2}\frac{x^{2m+2}y^2}{2!} - \cdots + x^{mn+m+n+1}y^{n+1}\right]z$$

$$\text{So} \quad x^{mn} = x^{mn+1}\begin{bmatrix} x^{m-1}y - x^{-3}\dfrac{x^{2m+2}y^2}{2!} + \cdots - x^{n-1}y^n + \\ \left[x^{n-1} - x^m y + x^{m-3}\dfrac{x^{2m+2}y^2}{2!} - \cdots + x^{m+n}y^{n+1}\right]z \end{bmatrix}$$

Thus $x^{mn} = x^{mn+1}y$,

where

$$y = x^{m-1}y - x^{-3}\frac{x^{2m+2}y^2}{2!} + \cdots - x^{n-1}y^n + \left(x^{n-1} - x^m y + x^{m-3}\frac{x^{2m+2}y^2}{2!} - \cdots + x^{m+n}y^{n+1}\right)z$$

Therefore R is a right mn-regular .■

Proposition 2.9:

Let R be a ring in which every maximal right ideal is m-regular. Then R is right non-singular ring if $r(a^m) \subset r(a)$ for all $a \in R$ and a fixed positive integer $m \neq 1$.

Proof:

If $Y(R) \neq 0$, then there exists $0 \neq a \in Y(R)$ such that $a^2 = 0$. First suppose that $aR + r(a) \neq R$. Thus, there is a maximal ideal M such that $aR + r(a) \subseteq M$ .Since M is right m-regular , then there exists b∈M and a fixed positive integer $m \neq 1$ such that $a^m = a^{m+1}b$. It follows that $a^m(1-ab) = 0$, that is $(1-ab) \in r(a^m) \subset r(a) \subset M$. Hence $1 \in M$, a contradiction. Therefore $aR + r(a) = R$. In particular, $ar + d = 1$ for some $r \in R$ and $d \in r(a)$.Then $a^2 r = a$. Thus $a = 0$, that is $Y(R) = 0$.■

Proposition 2.10 :

Let *R* be m-regular ring, then $J(R)$ is nilideal.

Proof:

Let $0 \neq a \in J(R)$, then $a^m \in J(R)$. Since R is m-regular, so there exists $c \in R$ such that $a^m = a^m c a^m$.

Hence $(1 - ca^m)$ is invertable, so there exists $u \in R$ such that $u(1 - ca^m) = 1$. It follows that $u(a^m - a^m c a^m) = a^m = 0$. Thus a is nilpotent element .Therefore J(R) is nilideal. ■

Corollary 2.11:

Let *R* be a reduced m-regular ring. Then $J(R) = (0)$.

Proof :

If $J(R) \neq (0)$, then there exists $a \in J(R)$ with $b \in R$ such that $a^m = a^m b a^m$, then $a^m - a^m b a^m = 0$.

Hence $a^m(1 - ba^m) = 0$. Since $a \in J(R)$, that is $a^m \in J(R)$ and $ba^m \in J(R)$, therefore $1 - ba^m$ is invertable.

Then there exists an invertable $u \in R$ such that $(1 - ba^m)u = 1$, implies that $(a^m - a^m b a^m)u = a^m$. Thus $a^m = 0$. Since *R* is reduced. Therefore $a = 0$.■

Preposition 2.12 :

Let *R* be semi-prime m-regular ring. Then the Center of *R* is right and left m-regular ring .

Proof :

Let $0 \neq a \in Cent(R)$, the Center of R, and let $a^2 = 0$, then $a^2 R = 0$, which gives $aRa = 0$. Since R is semi-prime, then a=0[6 p. 9.2.7]. Therefore $Cent(R)$ is reduced.

Now, let $c \in Cent(R)$,then there exists $b \in R$ and a fixed positive integer $m \neq 1$ such that $c^m = c^m bc^m$ (R is m-regular).If we set $d = c^{2m} b^3 \in Cent(R)$. Now,

$$c^{m+1}d = c^{m+1}c^{2m}b^3$$
$$= cc^m c^m c^m bbb = cc^m bc^m bc^m b$$
$$= cc^m bc^m b$$
$$= c^{m+1}b$$

Since R is m-regular , then every element is left and right m-regular, hence

$$c^{m+1}b = c^m$$

$$(c^m - c^{m+1}d)^2 = (c^m - c^{m+1}d)(c^m - c^{m+1}d)$$
$$= c^{2m} - c^{2m+1}d - c^{m+1}dc^m + (c^{m+1}d)(c^{m+1}d)$$
$$= c^{2m} - c^{2m+1}d - c^{m+1}dc^m + c^{m+1}dc^{m+1}d$$
$$= c^{2m} - c^m c^{m+1}d - c^{m+1}dc^m + c^{2m} = 0$$

Since $Cent(R)$ is reduced. Thus $c^m - c^{m+1}d = 0$

Then $c^m = c^{m+1}d$ and $c^m = dc^{m+1}$

Therefore $Cent(R)$ is right and left m-regular ring .■

Proposition 2.13:

Let I be any right ideal of a duo ring R. Then an element a of I is m-regular if and only if it is m-regular element in the ring R.

Proof:

Let a be m-regular element in I, and let b be any element of the ideal (a) generated by a in R. Then we have $b = na + ua + av + \sum u_i av_i$, where n is a positive integer and u and v are elements of R. Since a is m-regular element then there exists an element $x \in I$ such that $a^m = a^m xa^m$, $m \neq 1$ is a fixed positive integer. Consequently

$$b^m = [na + ua + av + \sum u_i av_i]^m$$
$$= [(na + ua) + (av + \sum u_i av_i)]^m$$
$$= (na + ua)^m + (na + ua)^{m-1}(av + \sum u_i av_i) + (na + ua)^{m-2}\frac{(av + \sum u_i av_i)^2}{2!} +$$
$$\cdots + (av + \sum u_i av_i)^m$$

Hence we have $b \in (a)'$, where $(a)'$ denotes an ideal generated by a in I. Therefore b is m-regular and the element a is m- regular element in R. The converse part is clear.■

Proposition 2.14:

A ring R is m-regular ring if and only if $r(a^m)$ is direct summand with every principal left ideal for a fixed integer $m \neq 1$.

Proof:

Suppose that $r(a^m) \oplus Ra^m = R$, for every a in R and a fixed positive integer $m \neq 1$. In particular $x + ba^m = 1$, then $a^m x + a^m ba^m = a^m$. S0 $a^m = a^m ba^m$ .Therefore R is m-regular.

Conversely: Assume that R is m-regular ,then for each a in R $a^m = a^m b a^m$ for some b in R, then $a^m(1-ba^m)=0$. So $(1-ba^m) \in r(a^m)$. Now, since $1 = ba^m + (1-ba^m)$ then $R = Ra^m + r(a^m)$. Now to prove $Ra^m \cap r(a^m) = 0$. Let $x \in Ra^m \cap r(a^m)$, then $x \in Ra^m$ and $a^m x = 0$ and so $x = ba^m$ for some b in R then $a^m b a^m = 0$. So $a^m = 0$. Therefore x=0.■

3- The Relation between m-Regular Ring and Other Rings

In this section we give the relation between m-regular rings and regular rings, reduced rings, local rings , $\pi - regular$ rings and uniform rings.

Proposition 3.1 :
Every reduced regular ring is left and right m-regular ring.

Proof :
Let R be a regular ring, and let $a \in R$, then there exists an element $b \in R$ such that $a = aba$,then $a - aba = 0$. It follows that $a(1-ba) = 0$, that is $(1-ba) \in r(a) = l(a) \subset l(a^m)$. Hence $(1-ba)a^m = 0$. So $a^m = ba^{m+1}$, that is R is left m-regular ring. Now, $(1-ab)a = 0$, implies that $(1-ab) \in l(a) = r(a) \subset r(a^m)$. Thus $a^m(1-ab) = 0$. So $a^m = a^{m+1}b$ .Therefore R is right m-regular ring.■

Corollary 3.2:
Let R be a ring whose maximal right ideals are right m-regular. Then R is right and left m-regular, if $r(a^m) \subset r(a)$ for all $a \in R$, and a fixed positive integer $m \neq 1$.

Proof:
Let $0 \neq a \in R$. We claim first $aR + r(a) = R$. If not, there exists a maximal right ideal M containing $aR + r(a)$. Since M is a right m-regular ideal, then there exists $b \in M$ such that $a^m = a^{m+1}b$. It follows that $a^m(1-ab) = 0$, that is $1 - ab \in r(a^m) \subset r(a)$, then $1 - ab \in r(a)$, since $a \in M$ then $ab \in M$ and so $1 \in M$,contradiction. Therefore $R = aR + r(a)$. In particular $1 = ar + d$ for some $r \in R$ and $d \in r(a)$. Hence $a = a^2 r + ad$ implies $a = a^2 r$ and then by Proposition (3.1) ,R is a right and left m-regular ring.■

Proposition 3.3:
Let R be a ring whose maximal right ideals are right m-regular. Then every right R-modules is p- injective if $r(a^m) \subset r(a)$., for all $a \in R$.

Proof:
By a similar method of proof used in Corollary (3.2) , we have $a = a^2 r$ for some r in R, then $a = ara$. Now, let $f : aR \to L$ be any right R- homomorphism, and let $f(ar) = y \in L$ [L is an R- module]. Then for any $c \in R$; $f(ac) = f(arac) = f(ar)ac = yac$. This means that every right R-module is p-injective.■

Lemma 3.4: [9]
If R is a right p-injective, then J(R) = Y(R).

Corollary 3.5:

Let R be m-regular ring. Then r(a) is essential in R for any a in R, if the set of non units elements is an ideal of R, with $r(a^m) \subset r(a)$ for a fixed positive integer $m \neq 1$.

Proof:

Let S be the set of non units element . Then S is contained in unique maximal ideal M by (p.158 in [11]), that is; J(R) is a unique maximal left ideal of R. Hence $Ra \neq R$ and $a \in J(R)$, and J(R) is m-regular ,that is J(R) is a right m-regular and hence by Proposition(3.3) R is p-injective module, which implies that J(R) =Y(R) by Lemma (3.4). So, $a \in Y(R)$, therefore r(a) is essential.∎

Recall that, a ring R is said to be uniform if all non zero- ideal of R is essential.

Recall that, a ring R is said to be local [6] if it has a unique maximal ideal.

Proposition 3.6:

Let R be a right m-regular ring, satisfies $r(a^m) \subset r(a)$ for all $a \in R$. Then R is local ring if and only if R is uniform ring.

Proof:

Let R be a right m-regular , if R is local, then for all non- zero element $a \in R$, aR essential . Now, if $aR \neq R$, then there exists a maximal ideal M such that $aR \subset M$ and since R is local ring , then M=J(R) that is $a \in J(R)$, then every ideal is right m-regular and by Proposition(3.3).That is R is right p-injective and by Lemma (3.4) ,we have $a \in Y(R)$, that is r(a) is essential for every $a \in R$ and hence R is uniform ring.

Conversely: Assume that R is uniform , that is r(a) is essential for every $a \in R$, and hence $a \in Y(R)$. Since R is right m-regular and by Proposition(3.3).That is R is right p-injective and by Lemma (3.4) $Y(R) = J(R)$. Thus $a \in J(R)$. Hence (1-a) is invertible. Therefore R is local ring by [6, Proposition 10.1.3].∎

Proposition 3.7:

Let R be a reversible ring. Then R is reduced ring if every maximal essential right ideal of R is right m-regular.

Proof :

Let $0 \neq a \in R$ such that $a^2 = 0$. If there exists a maximal right ideal M of R containing $r(a)$, then M must be an essential right ideal. Otherwise $M = r(e)$, $0 \neq e^2 = e \in R$, since R reversible, then $a \in M = r(e) = l(e)$ hence $ea = 0$ and we get $e \in l(a) = r(a) \subseteq M = r(e)$ that is $e^2 = 0$, contradiction. Hence M is essential and so M is right m-regular, then there exists $b \in M$ and an integer $m \neq 1$ such that $a^m = a^{m+1}b$.

It follows that $a^m(1-ab) = 0$, that is $1-ab \in r(a^m)$ since R is reversible. Then $r(a^m) = r(a)$, so $1-ab \in r(a) \subseteq M$, and we get $1 \in M$, contradiction. Therefore R is reduced.∎

Theorem 3.8:

Let R be local ring. Then R is m-regular if and only if R is π-regular ring with bounded index of nilpotency.

Proof :

Let R be m-regular ring. Then it is obvious that R is π -regular with bounded index of nilpotency.
Now, let $a \in R$, then if $aR \neq R$, then there exists a maximal ideal M such that $aR \subset M$ .Since R is local ring, then $M = J(R)$ that is $a \in J(R)$ and by Proposition (2.10), $a \in N(R)$, that is there exists a positive integer n such that $a^n = 0 = a^n ba^n$. But R has property bounded index of nilpotency. Therefore R is m-regular ring. Now, if $aR = R$ and $Ra = R$ (Since R is locally).
Then $ar = 1$ and $ca = 1$, for some $c, r \in R$
That is $a^2 r = a$ and $ca^2 = a$
Hence $a^m = a^{m+1} r$ and $a^m = ca^{m+1}$, for a fixed positive integer $m \neq 1$. That is R is right and left m-regular . Therefore R is m-regular.■

REFERENCES

[1] Birkenmeier, G.F., Kim, J.Y. and park, J.K. (1994), "A Connection Between Weak Regularity And The Simplicity Of Prime Factor Rings", American Math. Sco. 122(1), pp.53-58.

[2] Brown, S.H. (1973), "Rings Over Which Every Simple Modular Is Rationally Complete", canad. J. Math. 25 pp.693-701.

[3] Cohn , P.M. (1999), " Reversible Rings " Bull. London Math. Soc., 31, pp. 641-648.

[4] Fisher, J.W. and R.L. Snider (1974), "On The Von Neumann Regularity Of Rings With Prime Factor Rings" pacific Journal Math. 64, pp,135-144.

[5] Fuchs, L. and Rangaswamy, K.M. (1968), "On Generalized Regular Rings", Math. Zeitschr. 107, pp.71-81.

[6] Michiel Hazewinkel, Nadiya Gubareni and V.V. kirichenko, (2004), **"Algebras, Rings And Modules"** V.1 kuluwer Academic publishers**.**

[7] McCoy, N.H. (1939), "Generalized Regular Rings", Bull. Amer. Math. Soc. Vol. 45, pp 175-178.

[8] Neumann, J.V. (1936), "On Regular Rings" Princeton N. J. Vol. 22, pp 707-713.

[9] Nicholson, W.K. and Yousif, M.F. (1995), "Principally Injective Rings", J. Algebra, 174, pp. 77-93.

[10] Stenstrom, B. (1977), "Ring Of Quotient", Springer–Verlag, Berlin Heidelberg, New York.

[11] Thomas W. Hungerford (2004), "**An Introduction To Abstract Algebra**", Cleveland state university.

[12] Yue Chi. Ming , R. (1974), "On Von Neumann Regular Rings", proc, Edinburgh Math. Soc. 19, pp. 89-91.