

Thin Films Flow Driven on an Inclined Surface

Khider M.S. Khider

College of Computer Sciences and Mathematics
University of Mosul

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ABSTRACT

The flow of unsteady incompressible two dimensional system flow of a thin liquid films with negligible inertia is investigated. Continuity equation and Navier-Stokes equations are used to obtain the equation that governs this type of flow.

Keyword: Flow , Thin liquid films , Navier-Stokes equations

تدفق السطوح الرقيقة المدفوعة على سطح مائل

خضر محمد خضر

كلية علوم الحاسوب والرياضيات

جامعة الموصل

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الملخص

يهدف البحث إلى دراسة ميكانيكية الجريان اللا مستقر واللا منضغط للأغشية الرقيقة بانعدام قوى القص الذاتي وفي نظام ثنائي البعد، وقد استخدمت معادلة الاستمرارية ومعادلات نافير-ستوكس لإيجاد المعادلة التي تحكم هذا النوع من الجريان.

الكلمات المفتاحية: الجريان ، الأغشية الرقيقة، معادلات نافير - ستوكس

Introduction:

The flow of thin films of fluids is encountered in many engineering and biological applications. They include; the flow of rainwater on a road, windscreen or other draining problem [3], paint and coating flow [6, 1]. The flow of many protective biological fluids [5], and other coating are paint and dry processes [4, 7, 8]. The fluid film thickness and the average fluid flux are the main characteristics of interest in these applications [9]. Bascom, Cottington, and Singleterry [10] reported experimental observations of contact lines of thin liquid films. Emilia Borsa had studied the flow of a thin layer on a horizontal plate in the lubrication approximation[2]. The objective here is to obtain the equation governing the flow in thin liquid films, and to find the thickness of the film.

Governing Equations:

We consider a two-dimensional thin film flow on an inclined plane at angle α . The x-axis is oriented stream wise along the plane. The y-axis is perpendicular to the plane in the film thickness direction with the origin at the liquid plane interface. The flow is considered to be a laminar incompressible Newtonian fluid with constant density

ρ and constant viscosity μ , and governed by the Navier-Stokes equations and continuity equation as:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \rho g_x \quad \dots(1)$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \rho g_y \quad \dots(2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \dots(3)$$

In the thin film lubrication approximation the Navier-Stokes equations read

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \rho g \sin(\alpha) \quad \dots(4)$$

$$0 = -\frac{\partial p}{\partial y} - \rho g \cos(\alpha) \quad \dots(5)$$

Where g is gravitational acceleration, $y=h(x,y)$ is the free surface, p is the pressure in the fluid, u and p depended on $X=(x,y,t)$, and t is the time.

To complete the problem formulation, the lubrication equations (3), (4) and (5) require the boundary conditions:

$$u = 0 \quad \text{on} \quad y = 0 \quad \dots(6)$$

$$\tau = \mu \frac{\partial u}{\partial y} \quad \text{on} \quad y = h \quad \dots(7)$$

$$p = p_0 \quad \text{on} \quad y = h \quad \dots(8)$$

Where p_0 is the atmospheric pressure in the air face.

$$v = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} \quad \text{on} \quad y = h \quad \dots(9)$$

Where the boundary conditions (6), (7), (8) and (9) represent the no-slip condition, the balance of tangential stress, the balance of normal stress and the kinematics condition respectively.

Integrating (4) with respect to y and using the boundary condition (7) we have:

$$\mu \frac{\partial u}{\partial y} = \frac{\partial p}{\partial x} y - \rho g \sin(\alpha) y + \tau - \frac{\partial p}{\partial x} h + \rho g \sin(\alpha) h \quad \dots(10)$$

Similarly integrating (10) with respect to y and using boundary condition (6) we get :

$$u = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + \frac{\tau}{\mu} y - \frac{\rho g}{2\mu} \sin(\alpha) y^2 - \frac{1}{\mu} \frac{\partial p}{\partial x} h y + \frac{\rho g}{\mu} \sin(\alpha) h y \quad \dots(11)$$

Now, integrating (5) with respect to y and using the boundary condition (8) we obtain :

$$p = -\rho g \cos(\alpha) y + p_0 + \rho g \cos(\alpha) h \quad \dots(12)$$

Drive(12) with respect to x and substitution in to(11) we have:

$$u = \frac{\rho g}{2\mu} \cos(\alpha) \frac{\partial h}{\partial x} y^2 - \frac{\rho g}{\mu} \sin(\alpha) y^2 + \left(\frac{\tau}{\mu} - \frac{\rho g}{\mu} \cos(\alpha) \frac{\partial h}{\partial x} h + \frac{\rho g}{\mu} \sin(\alpha) h \right) y \quad \dots(13)$$

By using the continuity equation in the thin film approximation and the kinematics boundary condition leads the evolution equation for $y = h(x, t)$

$$\frac{\rho g}{2\mu} \cos(\alpha) \frac{\partial}{\partial x} \left(h^3 \frac{\partial h}{\partial x} \right) - \frac{3\rho g}{2\mu} \sin(\alpha) \frac{\partial h}{\partial x} h^2 = \frac{\partial h}{\partial t} + \frac{\tau}{\mu} \frac{\partial h}{\partial x} h \quad \dots(14)$$

We introduce the following nondimensional variables defined by:

$$\bar{h} = \frac{h}{h_0}, \quad \bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{tU}{h_0}, \quad \bar{\tau} = \frac{L\tau}{\sigma} \quad \dots(15)$$

Where the velocity U and the length scale L are characteristic quantities of the problem, assume that $\delta = h_0/L$, where h_0 is the characteristic length for the film thickness.

Then, convert the equation (14) into no dimensional form in terms of the no dimensional variables \bar{h} , \bar{x} , \bar{t} , $\bar{\tau}$ and the equation (14) becomes

$$Bo \cos(\alpha) \frac{\partial}{\partial \bar{x}} \left(\bar{h}^3 \frac{\partial \bar{h}}{\partial \bar{x}} \right) = \frac{\partial \bar{h}}{\partial \bar{t}} + \frac{3}{2} \sin(\alpha) k \bar{h}^2 \frac{\partial \bar{h}}{\partial \bar{x}} + Ca \bar{\tau} \bar{h} \frac{\partial \bar{h}}{\partial \bar{x}} \quad \dots(16)$$

$$\text{Where } Bo = \delta^4 \frac{\rho g l^2}{3\mu U}, \quad Ca = \delta^2 \frac{\sigma_0}{\mu U}, \quad k = \delta \frac{\rho g h_0^2}{\mu U}$$

For the sake of simpler notation, we drop the "dash" from the non dimensional variables \bar{h} , \bar{x} , \bar{t} , $\bar{\tau}$ in the equation (16) and we take the unsteady flow the equation (16) and divided on $Bo \cos(\alpha)$, we get:

$$\frac{d}{dx} \left(h^3 \frac{\partial h}{\partial x} \right) = \frac{3}{2} H \tan(\alpha) h^2 \frac{\partial h}{\partial x} + M \sec(\alpha) \tau h \frac{\partial h}{\partial x} \quad \dots(17)$$

$$\text{where } H = \frac{k}{Bo}, \quad M = \frac{Ca}{Bo}$$

Integrating (17) with respect to x and divide on h^3 we obtain:

$$\frac{\partial h}{\partial x} = \frac{3}{2} H \tan(\alpha) + M \sec(\alpha) \tau \frac{1}{h} + \frac{A}{h^3} \quad \dots(18)$$

Where A is constant.

Now, we take two cases:

The first case when $A = 0$ and $\tau = 0$ the equation (18) becomes

$$\frac{\partial h}{\partial x} = \frac{3}{2} H \tan(\alpha) \quad \dots(19)a$$

Integrating (19)a with respect to x we have:

$$h(x) = \frac{3}{2} H \tan(\alpha) x + f \quad \dots(20)a$$

f is constant.

By giving different value to the constant f, and angle α , we get the thick the film.

The second case when $A \neq 0$ and $\tau = 0$, the equation (18) becomes:

$$\frac{\partial h}{\partial x} = \frac{3}{2} H \tan(\alpha) + \frac{A}{h^3} \quad \dots(19)b$$

Now, to get the initial condition for the equation (19)b, we suppose that

$$\frac{\partial h}{\partial x} = 0 \text{ and the equation (19)b becomes:}$$

$$h^3(x) = \frac{-A}{\frac{3}{2} H \tan(\alpha)} \quad \dots(20)b$$

By taking cubic root the equation (26)b, we have:

$$h(x) = \sqrt[3]{\frac{-A}{\frac{3}{2}H \tan(\alpha)}} \quad \dots(21)$$

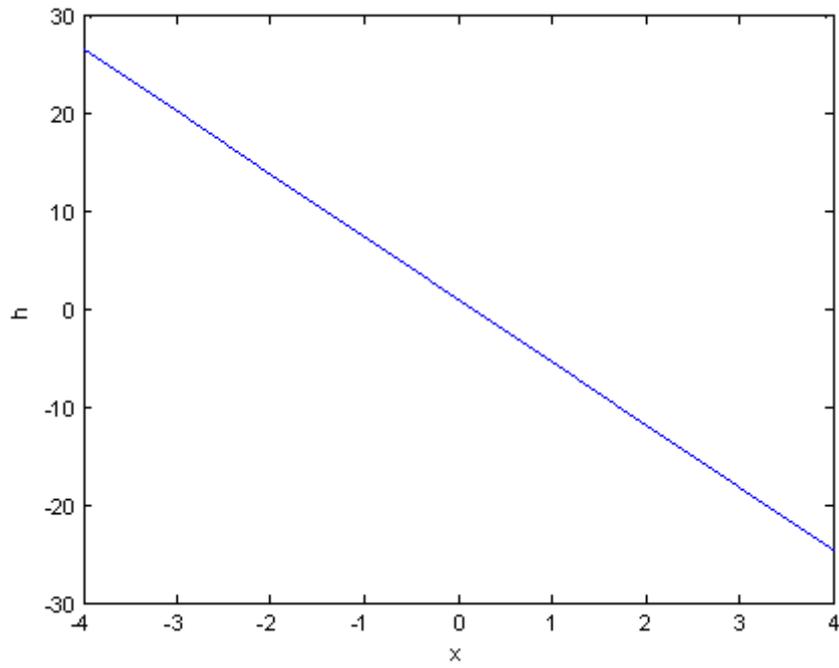
Similarly by taking different value for A and α we get the thick film in another case.

Table (1.1). Represent Solutions of Equation (20)a for Different α

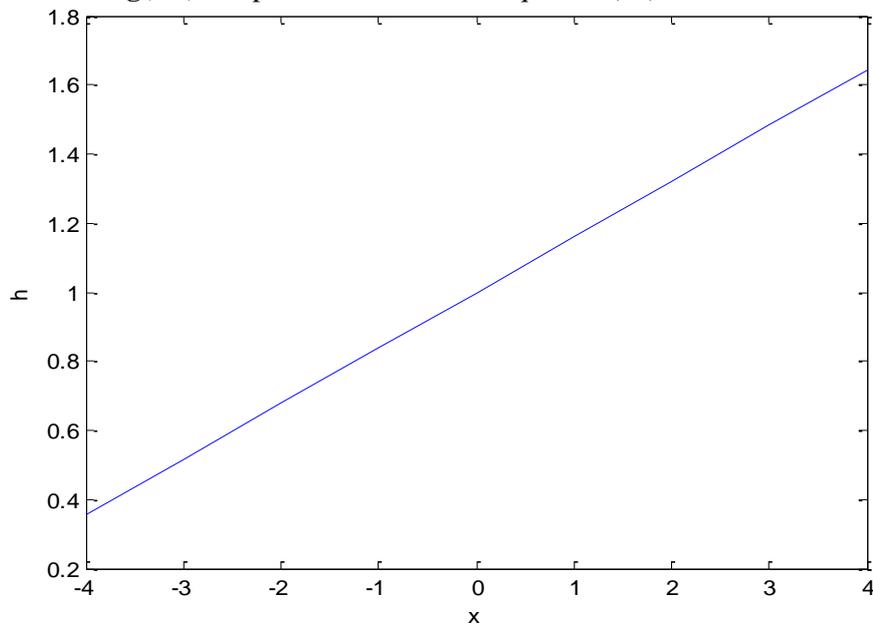
x	$\alpha = 30$	$\alpha = 41$	$\alpha = 49$	$\alpha = 60$
4.0	-24.6213	1.6426	-11.6916	2.2802
3.50	-21.4187	1.5623	-10.1052	2.1201
3.00	-18.2160	1.4820	-8.5187	1.9601
2.50	-15.0133	1.4016	-6.9323	1.8001
2.00	-11.8107	1.3213	-5.3458	1.6401
1.50	-8.6080	1.2410	-3.7594	1.4801
1.00	-5.4053	1.1607	-2.1729	1.3200
0.50	-2.2027	1.0803	-0.5865	1.1600
0	1.0000	1.0000	1.0000	1.0000
-0.50	4.2027	0.9197	2.5865	0.8400
-1.00	7.4053	0.8393	4.1729	0.6800
-1.50	10.6080	0.7590	5.7594	0.5199
-2.00	13.8107	0.6787	7.3458	0.3599
-2.50	17.0133	0.5984	8.9323	0.1999
-3.00	20.2160	0.5180	10.5187	0.0399
-3.50	23.4187	0.4377	12.1052	-0.1201
-4.00	26.6213	0.3574	13.6916	-0.2802

Table (1.2). Represent Solutions of Equation (21) for Different α

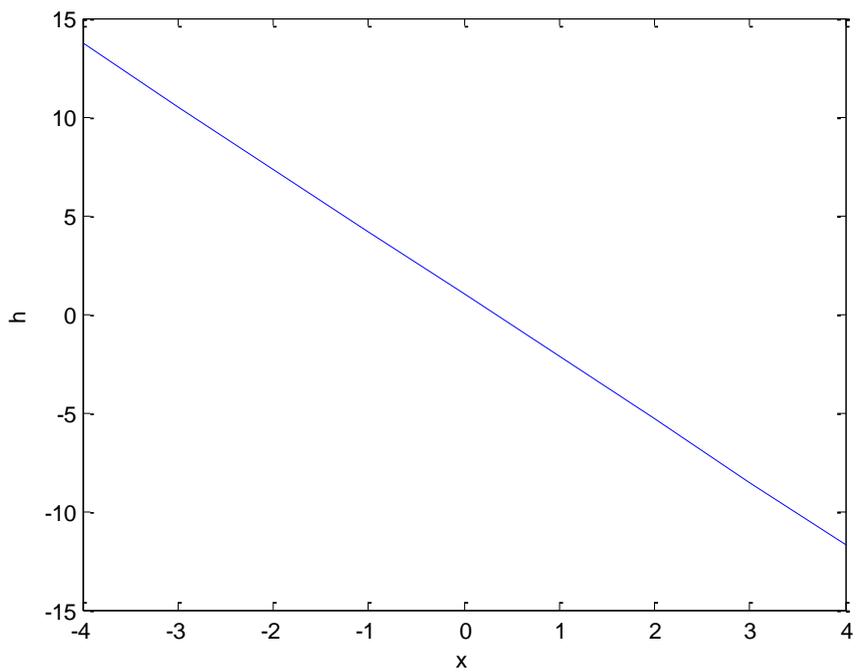
α	30	40	50	60
f	0.2137	0.3825	0.6126	1.1603



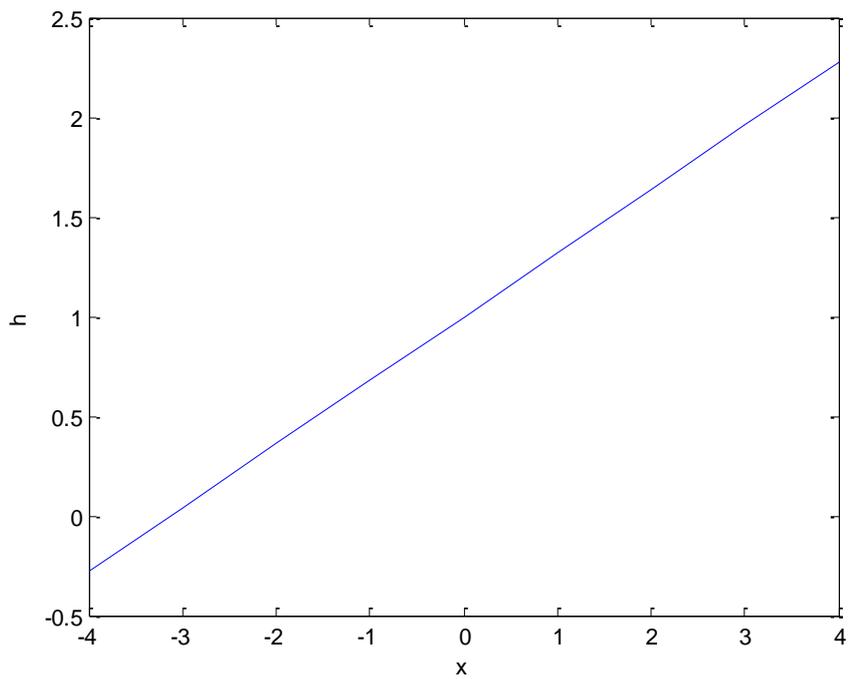
Fig(1.1). Represent Solutions of Equation (20) $a=1, \alpha = 30$



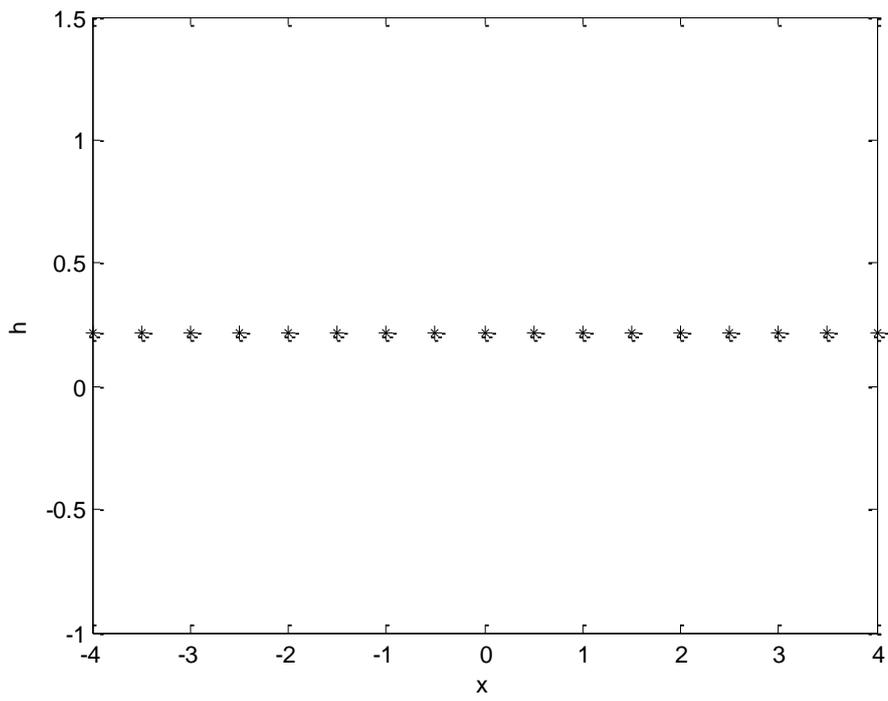
Fig(1.2). Represent Solutions of Equation (20) $a=1, \alpha = 41$



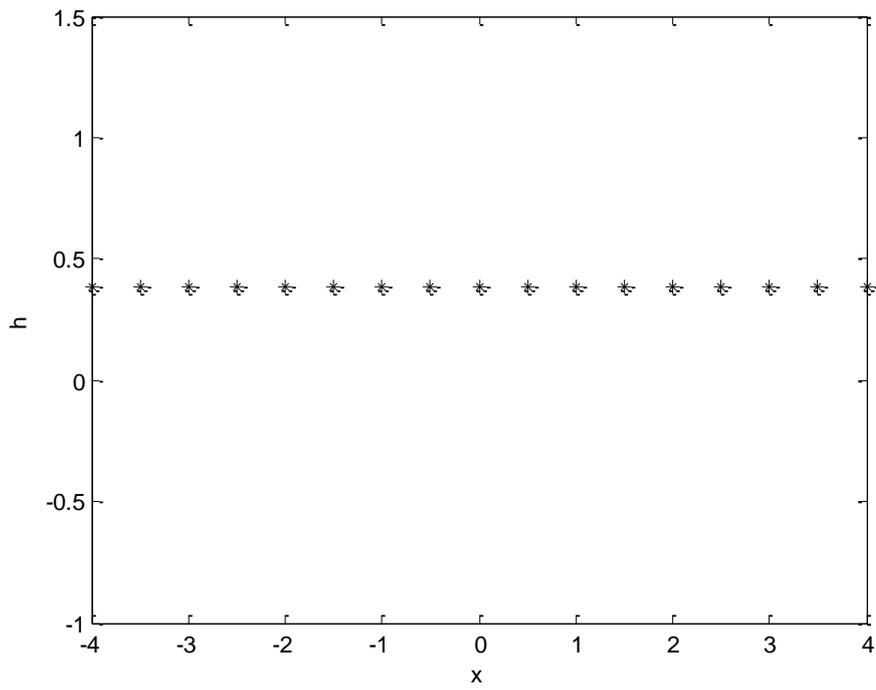
Fig(1.3). Represent Solutions of Equation (20)a $f=1, \alpha = 49$



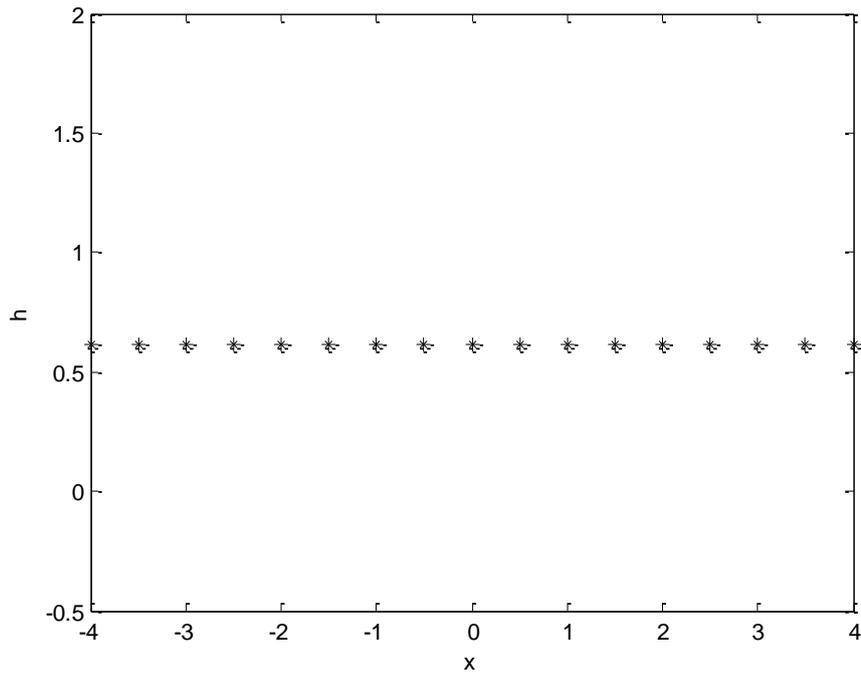
Fig(1.4). Represent Solutions of Equation (20)a $f=1, \alpha = 60$



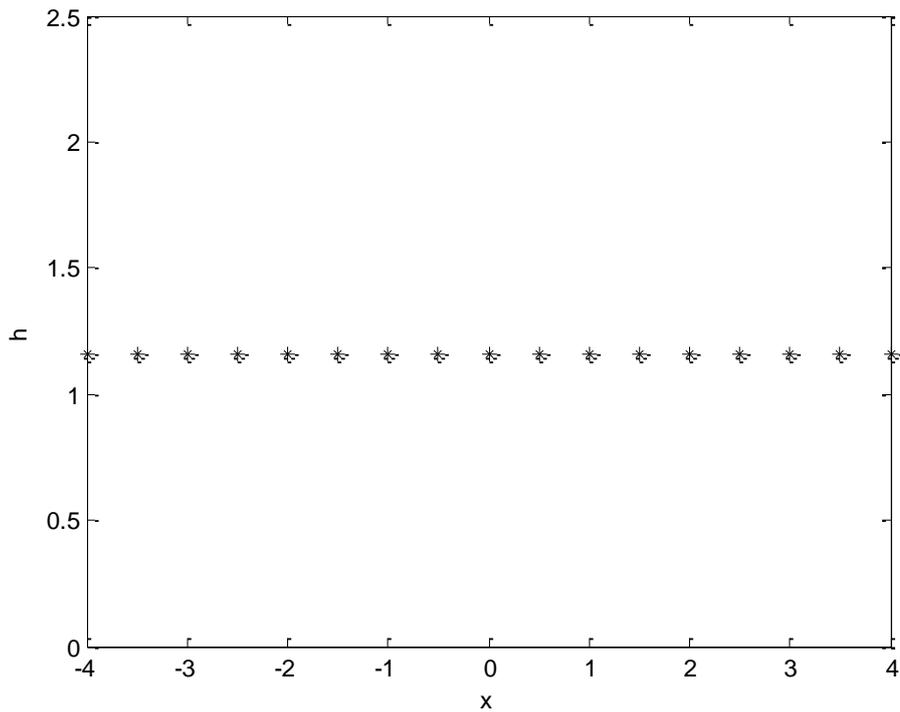
Fig(1.5). Represent Solutions of Equation (21) $\alpha = 30$



Fig(1.6). Represent Solutions of Equation (21) $\alpha = 40$



Fig(1.7). Represent Solutions of Equation (21) $\alpha = 50$



Fig(1.8). Represent Solutions of Equation (21) $\alpha = 60$

Conclusion:

Through our studies to the motion equation for viscous incompressible liquid, we conclude from equation (20)a that the thickness of the film increases when we approach the negative values of x when $\alpha=30,49$ as shown in Figures (1.1) and (1.3) and

it will decrease towards the positive values of x , when $\alpha=41, 60$ as shown in Figures (1.2) and (1.4) and the table (1.1), this implies that the value of the angle α will affect the thickness of the film. From equation (21), we note that the thickness of the film will be parallel to the x -axis and it will increase according to the value of α as shown in Figures (1.5), (1.6), (1.7) and (1.8) and the table (1.2).

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