Existence and Uniqueness of Solutions for Certain Nonlinear Mixed Type Integral and Integro-Differential Equations

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ABSTRACT

The aim of this paper is to study the existence, uniqueness and other properties of solutions of certain Volterra-Fredholm integral and integro differential equations. The tools employed in the analysis are based on the applications of the Banach fixed point theorem coupled with Bielecki type norm and certain integral inequalities with explicit estimates.

Keywords: Existence and uniqueness of solutions, mixed Volterra-Fredholm type, Banach fixed point theorem, integral inequalities, Bielcke type norm, continuous dependence.

وجود وحدانية الحلول لمعادلات تكاملية ومعادلات تكاملية – تفاضلية خاصة مختلطة

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الملخص

الهدف من هذا البحث هو دراسة الوجود والوحدانية وخواص أخرى لحلول المعادلة التكاملية من نوع فولتيرا – فريدهولم والمعادلات التكاملية – التفاضلية. التقنيات المستخدمة في التحليل تعتمد على تطبيقات مبرهنة بناخ للنقطة الثابتة مزدوجة مع النظيم من نوع بيلسكي والمتباينات التكاملية الخاصة مع التقديرات الصريحة.

الكلمات المفتاحية: وجود ووحدانية الحلول ، نوع فولتيرا – فريدهولم ، مبرهنة بناخ للنقطة الثابتة،متباينات تكاملية، مقياس من نوع بيلكسي ، الاعتماد المستمر.

1. Introduction

Consider the nonlinear Volterra-Fredholm integral and integro differential equations of the form.

$$x(t) = f(t, x(t), \int_a^t k(t,\sigma,x(\sigma))d\sigma, \int_a^b h(t,\sigma,x(\sigma))d\sigma), \qquad \ldots(1.1)$$

and

$$x'(t) = f(t, x(t), \int_a^t k(t,\sigma,x(\sigma))d\sigma, \int_a^b h(t,\sigma,x(\sigma))d\sigma), \qquad \ldots(1.2)$$

$$x(a) = x_0$$

for $-\infty < a \le t \le b < +\infty$, where x, f, k, h are real vectors with n components and ▾ denotes the derivative.

Let R^n denotes the real n-dimensional Euclidean space with appropriate norm denoted by $|.|$ and R the set of real numbers. Let $I = [a, +\infty)$, $R_+ = [0, \infty)$, be the given subset of R and assume that $k, h \in c(I^2 \times R^n, R^n)$ for $a \le s \le t \le b < +\infty$, $f \in c(I \times R^n \times R^n \times R^n, R^n)$.

Integral and integro differential equations arise in a variety of applications their study is of great interest .Many authers studied the equations of the forms (1.1) and (1.2) and their special and general versions with different view points, (see [1, 2, 3, 5, 7, 8]). The purpose of this paper is to study the existence, uniqueness and other properties of solutions of equations (1.1) and (1.2) under various assumptions on the functions f, k and h. The main tools employed in the analysis are based on the application the Banach fixed point theorem, coupled with Bielcke type norm and the integral inequalities with explicit estimates given in [6] .

2. Existence and Uniqueness

We first construct the appropriate metric space for our analysis [4]. Let $\beta > 0$ be a constant and consider the space of continuous functions $c(I, R^n)$ such that $\sup|x(t)| \big/ e^{\beta(t-a)} < \infty$, and denote this special space by $c_\beta(I, R^n)$. We couple the linear space $c_\beta(I, R^n)$ with suitable metric,

$$d_\beta^\infty(x, y) = \sup_{t\in I} \frac{|x(t) - y(t)|}{e^{\beta(t-a)}}$$

and a norm defined by

$$|x|_\beta^\infty = \sup_{t\in I} \frac{|x(t)|}{e^{\beta(t-a)}}$$

The above definitions of d_β^∞ and $|.|_\beta^\infty$ are variants of Bieleckis metric and norm [6].

The following Lemma proved in [4] deals with some important properties of d_β^∞ and $|.|_\beta^\infty$.

Lemma (2.1): If $\beta > 0$ is a constant, then:

i) d_β^∞ is a metric space,

ii) $|.|_\beta^\infty$ is a norm,

iii) $(C_\beta(I, R^n), |.|_\beta^\infty)$ is a Banach space,

iv) $(C_\beta(I, R^n), d_\beta^\infty)$ is a complete metric space.

Our main results concerning the existence and uniqueness of solutions of equations (1.1) and (1.2) are given in the following theorems .

Theorem (2.1):

Let $L, N > 0,\ M \geq 0, \beta > 0, \gamma > 1$ be constants with $\beta = (L + N^*)\gamma$, suppose that the functions f, k, h in equation (1.1) satisfy the conditions

$$|f(t,u,v,w) - f(t,\bar{u},\bar{v},\bar{w})| \leq M\{|u-\bar{u}| + |v-\bar{v}| + |w-\bar{w}|\} \quad \ldots(2.1)$$

$$|k(t,s,u) - k(t,s,\bar{u})| \leq L|u-\bar{u}|$$

$$|h(t,s,v) - h(t,s,\bar{v})| \leq N|v-\bar{v}| \quad \ldots(2.2)$$

and

$$d_1 = \sup_{t\in I} \frac{1}{e^{\beta(t-a)}} \left| f(t,0,\int_a^t k(t,\sigma,0)d\sigma, \int_a^b h(t,\sigma,0)d\sigma \right| < \infty$$

If $M(1+\frac{1}{\gamma})<1$, then the integral equation (1.1) has a unique solution $x \in C_{\beta}(I, R^n)$.

Proof:

Consider the following equivalent formulation of equation (1.1),

$$x(t) = f(t, x(t), \int_a^t k(t,\sigma,x(\sigma))d\sigma, \int_a^b h(t,\sigma,x(\sigma))d\sigma) -$$
$$- f(t,0,\int_a^t k(t,\sigma,0)d\sigma, \int_a^b h(t,\sigma,0)d\sigma) + f(t,0,\int_a^t k(t,\sigma,0)d\sigma, \int_a^b h(t,\sigma,0)d\sigma) \tag{2.3}$$

For $t \in I$, we will show that (2.3) has a unique solution and thus equation (1.1) must also have a unique solution.

Let $x \in C_{\beta}(I, R^n)$ and define the operator T by

$$(Tx)(t) = f(t, x(t), \int_a^t k(t,\sigma,x(\sigma))d\sigma, \int_a^b h(t,\sigma,x(\sigma))d\sigma) -$$
$$- f(t,0,\int_a^t k(t,\sigma,0)d\sigma, \int_a^b h(t,\sigma,0)d\sigma) + f(t,0,\int_a^t k(t,\sigma,0)d\sigma, \int_a^b h(t,\sigma,0)d\sigma) \tag{2.4}$$

Now, we shall show that T maps $C_{\beta}(I, R^n)$ into itself. From (2.4) and using the hypotheses, we have

$$|Tx|_{\beta}^{\infty} = \sup_{t\in I} \frac{|(Tx)(t)|}{e^{\beta(t-a)}}$$
$$\leq \sup_{t\in I} \frac{1}{e^{\beta(t-a)}} \Bigg| f(t, x(t), \int_a^t k(t,\sigma,x(\sigma))d\sigma, \int_a^b h(t,\sigma,x(\sigma))d\sigma) -$$
$$- f(t,0,\int_a^t k(t,\sigma,0)d\sigma, \int_a^b h(t,\sigma,0)d\sigma) \Bigg| +$$
$$+ \sup_{t\in I} \frac{1}{e^{\beta(t-a)}} \left| f(t,0,\int_a^t k(t,\sigma,0)d\sigma, \int_a^b h(t,\sigma,0)d\sigma) \right|$$
$$\leq d_1 + \sup_{t\in I} \frac{1}{e^{\beta(t-a)}} M \left\{ |x(t)| + \int_a^t L|x(\sigma)|d\sigma + \int_a^b N|x(\sigma)|d\sigma \right\}$$
$$= d_1 + M \Bigg\{ \sup_{t\in I} \frac{x(t)}{e^{\beta(t-a)}} + L \sup_{t\in I} \frac{1}{e^{\beta(t-a)}} \int_a^t e^{\beta(\sigma-a)} \frac{|x(\sigma)|}{e^{\beta(\sigma-a)}} d\sigma +$$
$$+ N \sup_{t\in I} \frac{1}{e^{\beta(t-a)}} \int_a^b e^{\beta(\sigma-a)} \frac{|x(\sigma)|}{e^{\beta(\sigma-a)}} d\sigma \Bigg\}$$

$$\leq d_1 + M\left\{|x|_\beta^\infty + L|x|_\beta^\infty \sup_{t\in I}\frac{1}{e^{\beta(t-a)}}\int_a^t e^{\beta(\sigma-a)}d\sigma + \right.$$

$$\left. + N|x|_\beta^\infty \sup_{t\in I}\frac{1}{e^{\beta(t-a)}}\int_a^b e^{\beta(\sigma-a)}d\sigma \right\}$$

$$= d_1 + M|x|_\beta^\infty\left\{1 + L\sup_{t\in I}\frac{1}{e^{\beta(t-a)}}\left(\frac{e^{\beta(t-a)}-1}{\beta}\right) + N\sup_{t\in I}\frac{1}{e^{\beta(t-a)}}\left(\frac{e^{\beta(b-a)}-1}{\beta}\right)\right\}$$

$$\leq d_1 + M|x|_\beta^\infty\left\{1 + L\sup_{t\in I}\frac{1}{e^{\beta(t-a)}}\left(\frac{e^{\beta(t-a)}-1}{\beta}\right) + N\frac{1}{e^{\beta(b-a)}}\left(\frac{e^{\beta(b-a)}-1}{\beta}\right)\right\}$$

$$= d_1 + M|x|_\beta^\infty\left\{1 + \frac{1}{\beta}(L+N^*)\right\} \quad where N^* = N\left(1 - \frac{1}{e^{\beta(b-a)}}\right)$$

$$= d_1 + M|x|_\beta^\infty\left\{1 + \frac{1}{\gamma}\right\} < \infty$$

This proves that the operator T maps $C_\beta(I,R^n)$ into itself. Now, we verify that the operator T is a contraction map.

Let $u,v\in C_\beta(I,R^n)$. From (2.4) and using the hypotheses we have

$$d_\beta^\infty(Tu,Tv) = \sup_{t\in I}\frac{|(Tu)(t)-(Tv)(t)|}{e^{\beta(t-a)}}$$

$$= \sup_{t\in I}\frac{1}{e^{\beta(t-a)}}\left| f(t,u(t),\int_a^t k(t,\sigma,u(\sigma))d\sigma,\int_a^b h(t,\sigma,u(\sigma))d\sigma) - \right.$$

$$\left. - f(t,v(t),\int_a^t k(t,\sigma,v(\sigma))d\sigma,\int_a^b h(t,\sigma,v(\sigma))d\sigma)\right|$$

$$\leq \sup_{t\in I}\frac{1}{e^{\beta(t-a)}}M\left\{ |u(t)-v(t)| + \int_a^t L|u(\sigma)-v(\sigma)|d\sigma + \int_a^b N|u(\sigma)-v(\sigma)|d\sigma\right\}$$

$$= M\left\{\sup_{t\in I}\frac{|u(t)-v(t)|}{e^{\beta(t-a)}} + \sup_{t\in I}\frac{1}{e^{\beta(t-a)}}L\int_a^t e^{\beta(\sigma-a)}\frac{|u(\sigma)-v(\sigma)|}{e^{\beta(\sigma-a)}}d\sigma + \right.$$

$$\left. + \sup_{t\in I}\frac{1}{e^{\beta(t-a)}}N\int_a^b e^{\beta(\sigma-a)}\frac{|u(\sigma)-v(\sigma)|}{e^{\beta(\sigma-a)}}d\sigma\right\}$$

$$\leq M\left\{d_\beta^\infty(u,v) + Ld_\beta^\infty(u,v)\sup_{t\in I}\frac{1}{e^{\beta(t-a)}}\int_a^t e^{\beta(\sigma-a)}d\sigma + \right.$$

$$\left. + Nd_\beta^\infty(u,v)\sup_{t\in I}\frac{1}{e^{\beta(t-a)}}\int_a^b e^{\beta(\sigma-a)}d\sigma\right\}$$

$$= Md_\beta^\infty(u,v)\left\{1 + L\sup_{t\in I}\frac{1}{e^{\beta(t-a)}}\left(\frac{e^{\beta(t-a)}-1}{\beta}\right) + N\sup_{t\in I}\frac{1}{e^{\beta(t-a)}}\left(\frac{e^{\beta(b-a)}-1}{\beta}\right)\right\}$$

$$\leq Md_{\beta}^{\infty}(u,v)\left\{1+L\sup_{t\in I}\frac{1}{e^{\beta(t-a)}}(\frac{e^{\beta(t-a)}-1}{\beta})+N\frac{1}{e^{\beta(b-a)}}(\frac{e^{\beta(b-a)}-1}{\beta})\right\}$$

$$=Md_{\beta}^{\infty}(u,v)\left\{1+\frac{1}{\beta}(L+N^{*})\right\} where N^{*}=(N-\frac{1}{e^{\beta(b-a)}})$$

$$=M(1+\frac{1}{\gamma})d_{\beta}^{\infty}(u,v)$$

Since, $M(1+\frac{1}{\gamma})<1$, it follows from the Banach fixed point theorem see[4] that T has a unique fixed point in $C_{\beta}(I,R^{n})$. The fixed point of T is, however, a solution of equation (1.1) .

Theorem (2.2):

Let L,N,β,M,γ be as in theorem (2.1) .Suppose that the functions f,k,h in equation (1.2) satisfy the conditions (2.1) ,(2.2) and

$$d_{2}=\sup_{t\in I}\frac{1}{e^{\beta(t-a)}}\left|x_{0}+\int_{a}^{t} f(s,0,\int_{a}^{s}k(s,\sigma,0)d\sigma,\int_{a}^{b}h(s,\sigma,0)d\sigma)ds\right|<\infty$$

If $M(1+\frac{1}{\gamma})<1$, then the integro differential equation (1.2) has a unique solution $x\in C_{\beta}(I,R^{n})$.

Proof:

Let $x\in C_{\beta}(I,R^{n})$, and define the operator S by

$$(sx)(t)=x_{0}+\int_{a}^{t} f(s,x(s),\int_{a}^{s}k(s,\sigma,x(\sigma))d\sigma,\int_{a}^{b}h(s,\sigma,x(\sigma))d\sigma)ds-$$

$$-\int_{a}^{t} f(s,0,\int_{a}^{s}k(s,\sigma,0)d\sigma,\int_{a}^{b}h(s,\sigma,0)d\sigma)ds+\int_{a}^{t} f(s,0,\int_{a}^{s}k(s,\sigma,0)d\sigma,\int_{a}^{b}h(s,\sigma,0)d\sigma)ds,$$

for $t\in I$, the proof that S maps $C_{\beta}(I,R^{n})$ into itself and is a contraction map, can be completed by closely looking at the proof of theorem (2.1) given a above with suitable modifications. Here, we omit the details .

3. Estimates on the Solutions

In this section, we obtain estimates of solutions of equations (1.1) and (1.2) under some suitable assumptions for the functions involved in them.

Lemma (3.1):

Let $u(t)\in C(I,R_{+}),r(t,s),l(t,s),\frac{\partial}{\partial t}r(t,s),\frac{\partial}{\partial t}l(t,s)\in C(I^{2},R_{+})$ be non decreasing in $t\in I$.

If

$$u(t)\leq c+\int_{a}^{t}r(t,s)u(s)ds+\int_{a}^{b}l(t,s)u(s)ds \qquad \ldots(3.1)$$

For $t\in I$ where $c\geq 0$ is a constant, then

$$u(t) \le B\exp(\int_a^t g(s)ds \qquad \ldots(3.2)$$

For $t \in I$, where

$$g(t) = r(t,t) + \int_a^t \frac{\partial}{\partial t} r(t,s)ds + \int_a^b \frac{\partial}{\partial t} l(t,s)ds \qquad \ldots(3.3)$$

and

$$B = c + \int_a^b \ l(a,s)u(s)ds \qquad \ldots(3.4)$$

Proof: Define a function $w(t)$ be the right hand side of (3.1). Then, $w(t) \ge 0$, $w(a) = B$. $u(t) \le w(t)$, $w(t)$ is non decreasing in t and

$$w'(t) = r(t,t)u(t) + \int_a^t \frac{\partial}{\partial t} r(t,s)u(s)ds + \int_a^b \frac{\partial}{\partial t} l(t,s)u(s)ds$$

$$\le (r(t,t) + \int_a^t \frac{\partial}{\partial t} r(t,s)ds + \int_a^b \frac{\partial}{\partial t} l(t,s)ds)u(s)$$

Let $g(t) = r(t,t) + \int_a^t \frac{\partial}{\partial t} r(t,s)ds + \int_a^b \frac{\partial}{\partial t} l(t,s)ds$

$$w'(t) \le g(t)u(t) \le g(t)w(t) \qquad \ldots(3.5)$$

The inequality (3.5) implies the estimate

$$w(t) \le B\exp(\int_a^t g(s)ds) \qquad \ldots(3.6)$$

Using (3.6) in $u(t) \le w(t)$ we get the required inequality (3.2).

Lemma (3.2): Let $u(t), p(t) \in C(I, R_+), r(t,s), l(t,s), \frac{\partial}{\partial t} r(t,s), \frac{\partial}{\partial t} l(t,s) \in C(I^2, R_+)$ be non decreasing in $t \in I$. If

$$u(t) \le c + \int_a^t p(s)[u(s) + \int_a^s r(s,\sigma)u(\sigma)d\sigma + \int_a^b l(s,\sigma)u(\sigma)d\sigma] \qquad \ldots(3.7)$$

For $t \in I$ where $c \ge 0$ is a constant, then

$$u(t) \le B[1 + \int_a^t p(s)\exp(\int_a^s [p(\sigma) + g(\sigma)]d\sigma)ds], \qquad \ldots(3.8)$$

For $t \in I$, where $g(t)$ and B is as in Lemma (3.1).

Proof: Define a function w(t) to be on the right hand side of (3.7). Then, w(t)≥0, w(a)=B, $u(t) \le w(t)$ *, and* $w(t)$ are non decreasing in *t*. Then,

$$w(t) = c + \int_a^t p(s)\left\{u(s) + \int_a^s r(s,\sigma)u(\sigma)d\sigma + \int_a^b l(s,\sigma)u(\sigma)u(\sigma)d\sigma \ \right\}ds$$

and

$$w'(t) = p(t)\left\{u(t) + \int_a^t r(t,\sigma)u(\sigma)d\sigma + \int_a^b l(t,\sigma)u(\sigma)u(\sigma)d\sigma \ \right\}$$

Put: $v(t) = u(t) + \int_a^t r(t,\sigma)u(\sigma)d\sigma + \int_a^b l(t,\sigma)u(\sigma)u(\sigma)d\sigma$

Then,

$$w'(t) \le p(t)v(t)$$

$$v'(t) = u'(t) + r(t,t)u(t) + \int_a^t \frac{\partial r(t,\sigma)}{\partial t} u(\sigma)d\sigma + \int_a^b \frac{\partial l(t,\sigma)}{\partial t} u(\sigma)d\sigma$$

$$v'(t) \le p(t)v(t) + v(t)\left\{ r(t,t) + \int_a^t \frac{\partial r(t,\sigma)}{\partial t} d\sigma + \int_a^b \frac{\partial l(t,\sigma)}{\partial t} d\sigma \right\}$$

$$v'(t) \le p(t)v(t) + g(t)v(t) \qquad \dots(3.9)$$

The inequality (3.9) implies the estimate

$$v(t) \le B\exp\left(\int_a^t p(s) + g(s)ds\right) \qquad \dots(3.10)$$

Using (3.10) in u(t) ≤ v(t), we get the required inequality (3.8).

First, we give the following theorem concerning the estimate on the solution of equation (1.1).

Theorem (3.1): Suppose that the functions f, k, h in equation (1.1) satisfy the conditions

$$\left| f(t,u,v,w) - f(t,\overline{u},\overline{v},\overline{w}) \right| \le A\left\{ |u-\overline{u}| + |v-\overline{v}| + |w-\overline{w}| \right\} \qquad \dots(3.11)$$

$$\begin{aligned} |k(t,\sigma,u) - k(t,\sigma,v)| &\le r(t,\sigma)|u-v| \\ |h(t,\sigma,u) - h(t,\sigma,v)| &\le l(t,\sigma)|u-v| \end{aligned} \qquad \dots(3.12)$$

Where, $0 \le A < 1$ is a constant and $r(t,\sigma), l(t,\sigma), \frac{\partial}{\partial t} r(t,\sigma), \frac{\partial}{\partial t} l(t,\sigma) \in C(I^2, R_+)$. Let $c_1 = \sup_{t\in I} \left| f(t,0,\int_a^t k(t,\sigma,0)d\sigma, \int_a^b h(t,\sigma,0)d\sigma) \right| < \infty$. If $x(t), t \in I$ is any solution of equation (1.1), then

$$|x(t)| \le \left(\frac{E_1}{1-A} \right) \exp\left(\int_a^t q(s)ds \right), \qquad \dots(3.13)$$

for $t \in I$, where $E_1 = c_1 + \int_a^b l(a,s)u(s)ds$ and

$$q(t) = \frac{A}{1-A} g(t) \qquad \dots(3.14)$$

in which g(t) is as defined in Lemma (3.1).

Proof: By using the fact that the solution $x(t)$ of equation (1.1) satisfies the equivalent equation (2.3) and the hypotheses we have

$$|x(t)| \le \left| f(t,0,\int_a^t k(t,\sigma,0)d\sigma, \int_a^b h(t,\sigma,0)d\sigma) \right| +$$

$$+ \left| f(t,x(t),\int_a^t k(t,\sigma,x(t))d\sigma, \int_a^b h(t,\sigma,x(t))d\sigma) - f(t,0,\int_a^t k(t,\sigma,0)d\sigma, \int_a^b h(t,\sigma,0)d\sigma) \right|$$

$$\leq c_1 + A\left\{ |x(t)| + \int_a^t r(t,\sigma)|x(t)|d\sigma + \int_a^b l(t,\sigma)|x(t)|d\sigma \right\} \qquad \ldots(3.15)$$

From (3.15) and using the assumptions $0 \leq A < 1$, we observe that

$$|x(t)| \leq \frac{c_1}{1-A} + \frac{A}{1-A}\int_a^t r(t,\sigma)|x(\sigma)|d\sigma + \frac{A}{1-A}\int_a^b l(t,\sigma)|x(\sigma)|d\sigma \qquad \ldots(3.16)$$

Now, an application of Lemma (3.1) to (3.16) yields (3.13).
Next, we shall obtain the estimate on the solution of equation (1.2).

Theorem (3.2)

Suppose that the function f in equation (1.2) satisfies the condition

$$|f(t,u,v,w) - f(t,\bar{u},\bar{v},\bar{w})| \leq p(t)\{|u-\bar{u}| + |v-\bar{v}| + |w-\bar{w}|\} \qquad \ldots(3.17)$$

Where $p \in C(I, R_+)$ and the functions of k, h in equation (1.2) satisfy the conditions (3.12). Let

$$c_2 = \sup_{t\in I}\left| x_0 + \int_a^t f(s,0,\int_a^s k(s,\sigma,0)d\sigma,\int_a^b h(s,\sigma,0)d\sigma)ds \right| < \infty$$

If $x(t)$, $t \in I$, is any solution of equation (1.2), then

$$|x(t)| \leq E_2\left\{ 1 + \int_a^t p(s)\exp(\int_a^s \{p(\sigma) + g(\sigma)\}d\sigma)ds \right\} \qquad \ldots(3.18)$$

For $t \in I$, where $E_2 = c_2 + \int_a^b l(a,s)u(s)ds \ and \ g(t)$ is as defined in Lemma (3.1) .

Proof:

Using the fact that $x(t)$ is a solution of equation (1.2) and that the hypotheses we have

$$|x(t)| \leq \left| x_o + \int_a^t f(s,0,\int_a^s k(s,\sigma,0)d\sigma,\int_a^b h(s,\sigma,0)d\sigma)ds \right| +$$

$$+\int_a^t \Big| f(s,x(s),\int_a^s k(s,\sigma,x(\sigma))d\sigma,\int_a^b h(s,\sigma,x(\sigma))d\sigma) -$$

$$- f(s,0,\int_a^s k(s,\sigma,0)d\sigma,\int_a^b h(s,\sigma,0)d\sigma) \Big| ds$$

$$\leq c_2 + \int_a^t p(s)\left\{ |x(s)| + \int_a^s r(s,\sigma)|x(\sigma)|d\sigma + \int_a^b l(s,\sigma)|x(\sigma)|d\sigma \right\}ds \qquad \ldots(3.19)$$

Now, an application of Lemma (3.2) to (3.19) yields (3.18) .

4. Continuous Dependence

In this section , we deal with the continuous dependence of solutions of equations (1.1) and (1.2) for functions involved in them. Consider the equations (1.1) and (1.2) and the corresponding equations

$$y(t) = \bar{f}(t, y(t), \int_a^t \bar{k}(t,\sigma,y(\sigma))d\sigma, \int_a^b \bar{h}(t,\sigma,y(\sigma))d\sigma), \qquad \ldots(4.1)$$

and

$$y'(t) = \bar{f}(t, y(t), \int_a^t \bar{k}(t,\sigma, y(\sigma))d\sigma, \int_a^b \bar{h}(t,\sigma, y(\sigma))d\sigma), \qquad \ldots(4.2)$$
$$y(a) = y_0$$

For $t \in I$, where $\bar{k}, \bar{h} \in C(I^2 \times R^n, R^n)$ for $a \le s \le t \le b < \infty$, *where* $\bar{f} \in C(I \times R^n \times R^n \times R^n, R^n)$.

The following theorems deal with the continuous dependence of solutions of equations (1.1) and (1.2) for functions involved in them.

Theorem (4.1)

Suppose that the functions f, k, h in equation (1.1) satisfy the conditions (3.11) and (3.12). Furthermore, suppose that

$$\Big| f(t, y(t), \int_a^t k(t,\sigma, y(\sigma))d\sigma, \int_a^b h(t,\sigma, y(\sigma))d\sigma) -$$
$$- \bar{f}(t, y(t), \int_a^t \bar{k}(t,\sigma, y(\sigma))d\sigma, \int_a^b \bar{h}(t,\sigma, y(\sigma))d\sigma) \Big| \le \varepsilon_1$$

Where f, k, h and $\bar{f}, \bar{k}, \bar{h}$ are the functions involved in equation (1.1) and (4.1), $\varepsilon_1 > 0$ is an arbitrary small constant and $y(t)$ is a solution of equation (4.1). Then, the solution $x(t)$, $t \in I$ of equation (1.1) depends continuously on the functions involved on the right hand side of equation (1.1).

Proof :

Let $u(t) = |x(t) - y(t)|, t \in I$, using the facts that $x(t)$ and $y(t)$ are the solutions of equations (1.1) and (4.1) and the hypotheses we have

$$u(t) \le \Big| f(t, x(t), \int_a^t k(t,\sigma, x(\sigma))d\sigma, \int_a^b h(t,\sigma, x(\sigma))d\sigma) -$$
$$- f(t, y(t), \int_a^t k(t,\sigma, y(\sigma))d\sigma, \int_a^b h(t,\sigma, y(\sigma))d\sigma \Big| +$$
$$+ \Big| f(t, y(t), \int_a^t k(t,\sigma, y(\sigma))d\sigma, \int_a^b h(t,\sigma, y(\sigma))d\sigma) -$$
$$- \bar{f}(t, y(t), \int_a^t \bar{k}(t,\sigma, y(\sigma))d\sigma, \int_a^b \bar{h}(t,\sigma, y(\sigma))d\sigma \Big|$$
$$u(t) \le \varepsilon_1 + A\left\{u(t) + \int_a^t r(t,\sigma)u(\sigma)d\sigma + \int_a^b l(t,\sigma)u(\sigma)d\sigma\right\} \qquad \ldots(4.3)$$

From (4.3) and using the assumption that $0 \le A < 1$, we observe that

$$u(t) \le \frac{\varepsilon_1}{1-A} + \frac{A}{1-A}\int_a^t r(t,\sigma)u(\sigma)d\sigma + \frac{A}{1-A}\int_a^b l(t,\sigma)u(\sigma)d\sigma \qquad \ldots(4.4)$$

Now, an application of Lemma (3.1) to (4.4) yields

$$|x(t) - y(t)| \le \left(\frac{E_3}{1-A}\right)\exp(\int_a^t q(s)ds) \qquad \ldots(4.5)$$

Where $q(t)$ is defined by (3.14) and $E_3 = \varepsilon_1 + \int_a^b l(a,s)u(s)ds$. From (4.5) it follows that the solution of equation (1.1) depends continuously on the functions involved on the right hand side of equation (1.1) .

Theorem (4.2)

Suppose that the functions f, k and h in equation (1.2) satisfy the conditions (3.17) and (3.12). Suppose that

$$|x_0 - y_0| + \int_a^t \Big| f(s, y(s), \int_a^s k(s,\sigma, y(\sigma))d\sigma, \int_a^b h(s,\sigma, y(\sigma))d\sigma) -$$
$$- \bar{f}(s, y(s), \int_a^s \bar{k}(s,\sigma, y(\sigma))d\sigma, \int_a^b \bar{h}(s,\sigma, y(\sigma))d\sigma) \Big| ds \le \varepsilon_2$$

Where f, k, h and $\bar{f}, \bar{k}, \bar{h}$ are the functions involved in equation (1.2) and (4.2), $\varepsilon_2 > 0$ is an arbitrary small constant and $y(t)$ is a solution of equation (4.2) . Then the solution $x(t)$, $t \in I$ of equation (1.2) depends continuously on the functions involved on the right hand side of equation (1.2).

Proof :

Let $u(t) = |x(t) - y(t)|$, $t \in I$, using the facts that $x(t)$ and $y(t)$ are the solutions of equations (1.2) and (4.2) and the hypotheses we have

$$u(t) \le |x_0 - y_0| + \int_a^t \Big| f(s, x(s), \int_a^s k(s,\sigma, x(\sigma))d\sigma, \int_a^b h(s,\sigma, x(\sigma))d\sigma) -$$
$$- f(s, y(s), \int_a^s k(s,\sigma, y(\sigma))d\sigma, \int_a^b h(s,\sigma, y(\sigma))d\sigma) \Big| ds +$$
$$+ \int_a^t \Big| f(s, y(s), \int_a^s k(s,\sigma, y(\sigma))d\sigma, \int_a^b h(s,\sigma, y(\sigma))d\sigma) -$$
$$- \bar{f}(s, y(s), \int_a^s \bar{k}(s,\sigma, y(\sigma))d\sigma, \int_a^b \bar{h}(s,\sigma, y(\sigma))d\sigma) \Big| ds$$
$$\le \varepsilon_2 + \int_a^t p(s) \left\{ u(s) + \int_a^t r(s,\sigma)u(\sigma)d\sigma + \int_a^b l(s,\sigma)u(\sigma)d\sigma \right\} ds \qquad \ldots(4.6)$$

Now, an application of Lemma (3.2) to (4.6) yields

$$|x(t) - y(t)| \le E_4 \left\{ 1 + \int_a^t p(s) \exp(\int_a^s \{p(\sigma) + g(\sigma)\} d\sigma) ds \right\} \qquad \ldots(4.7)$$

For $t \in I$, where $E_4 = \varepsilon_2 + \int_a^b l(a,s)u(s)ds$ and $g(t)$ is defined in Lemma (3.1). From (4.7), it follows that the solution of equation (1.2) depends continuously on the functions involved on the right hand side of equation (1.2).

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