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Modified Finite volume method for Solving Electrocardiography inverse problem

Ebtihal Sabah Al-Bayat
Ahmed Farooq Qasem
Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Mosul, Iraq.

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Correspondence:

Abstract

. In this paper we resorted to study finite volume method to solve inverse problem of heart using the Bidomain model of heart as a network of points to obtain approximate solutions, then using iteration method in nonlinear limit to improve the results obtained. From finite volume method to obtain more accurate results with a small error rate in order to determine the exact condition and perform medical intervention at correct time, solutions were obtained by relying on linear function as an initial condition in first case and non-linear function again in second case, and then improving these results in third case, We examined difference between results through tables and figures using the methods above mentioned. Results showed that approximate solutions were close enough to real one. Results also showed that proposed method by combining finite volume method and new iteration method is more accurate and efficient than classical finite volume method, as shown in tables andfigures.

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تحسين طريقة الحجوم المحددة لحل مسالة الاشارات الكهربائية للقلب

ابتهال صباح البياتي   احمد فاروق قاسم

كلية علوم الحاسوب والرياضيات جامعة الموصل

مستخلص البحث

اعتمدنا في هذا البحث على طريقة الحجوم المنتهية في حل المسالة العكسية لنظام المجال الثاني للقلب للحصول على حل تقريبي للنظام ومن ثم استخدام طريقة التكرار الجديدة في حل الحد غير الخطي في النظام لتحسين الحلول العددية والحصول على نسبة خطأ اقل، كذلك تم حل النظام بالاعتماد على شرط ابتدائي كحالة خطية مرة وحالة غير خطية مرة اخرى ومن ثم مقارنة النتائج لبيان كفاءة الطريقة المقترحة. وأظهرت النتائج أن الحلول التقريبية كانت قريبة بما فيه الكفاية من الحلول الحقيقية، كما أظهرت النتائج أن الطريقة المقترحة بالدمج بين طريقة الحجوم المنتهية وطريقة التكرار الجديدة أكثر دقة وكفاءة من طريقة الحجم المحدود الكلاسيكية كما مبين في الجداول والاشكال.

The mathematical equations simulating the potentials of the heart were developed directly from the microscopic model represented by cellular tissue as a periodic network of interconnected cells (for internal and external positions). To use these equations, we must have boundary conditions[1], These mathematical models provide flexibility for use in integrating clinical data or creating numerous experimental data by changing parameter values, noting that accurate simulation requires accurate and complete data to obtain comprehensive information about the patient's heart in order to carry out the correct clinical intervention at the appropriate time[2], Since the mechanics of the heart begins with contraction and coincides with the electrical wave in the right atrium and moves to the atria and ventricles, which may be irregular, medical intervention is necessary. To achieve this, the electrophysiology of the heart must be understood and the focus must be on errors in setting standards for the mechanical model of the heart[3], The inverse ECG problem is of great importance for the diagnosis and treatment of arrhythmia, so it has been studied using a Bidomain model imposing different constraints in order to obtain a numerical solution to reconstruct the electrical activity within the myocardium[4], Many phenomena are studied to control chaos, treat its stability, and then solve it using numerical methods [5, 6], Numerical simulation of the electrical activity of the heart has become widely used in recent years, as studies used to rely on representing tissues in a Monodomain model for re-entry and fibrillation, but the time has come to use a Bidomain model because it is better and easier to understand the heterogeneity of cardiac tissue because it has clinical importance and is more realistic[7], Solving partial differential equations by Bidomain model led to the development of the Monodomain approximation, which studies the difference between internal and external space, while the Bidomain model studies the effect of currents in the internal and external fields on the membrane potential and ionic currents across the membrane as well[8], The use of numerical methods to solve mathematical models is a valuable tool for understanding the disease and studying the effect of treatment, which is considered one of the basics of developing these models in providing clinical treatment by creating a mathematical model for other organs such as the stomach and the nervous system and using different numerical methods to solve them[9], When solving many of these problems and calculating the absolute error value for them, we rely on exact solutions and approximate solutions, but in many of them we do not have an accurate solution to compare the results, so we resort to choosing small values and additional iterations to obtain accurate results[10], At the same way as the finite element method, the finite volume method can be considered one of the numerical analysis techniques used to solve partial differential equations by considering separate locations in the form of a grid, converting them into algebraic equations, and then solving them to obtain the total volume and not just the points[11], One of the advantages of this method is that it maintains the numerical flow from one cell to another, that is, it maintains equilibrium by means of the divergence formula, and this is what makes it more widely used for modeling problems, especially in fluid mechanics, heat transfer, or petroleum engineering[12], It is possible to prove the efficiency of the solution method for the bidomain model of the heart, which depends on the amount of error for second-order or higher solutions[13]. Some physical phenomena are solved by a semi-analytical iterative method such as nonlinear second-order boundary value problems [14], The iteration method has attracted the attention of researchers in recent years, as it is considered a method with strong results when combined with algebraic computing programs to solve nonlinear partial differential equations[15,16,17].

1- BIDOMAIN MODEL

The Bidomain model is considered a mathematical method to express the electrical activity of the heart. It was created to simulate a heartbeat in order to study the changes occurring in the potential field at some locations of the patient’s torso, through which heart diseases such as arrhythmia can be diagnosed. [18]. The heart domain denoted by Ω and the conductivity tensor of intercellular and extracellular by σ_i ,σ_e ,let v_m be atransmembrane potential and u_i , u_e be electric potential of intercellular and extracellular respectively , such that $v_m = u_i - u_e$ .The bidomain model defined as:

$$\begin{cases} A_m\left(C_m \frac{\partial v_m}{\partial t} + I_{ion}(v_m, w)\right) - div(\sigma_i \nabla v_m) = div(\sigma_i \nabla u_e) & in\, \Omega \times]0, T[\\ div\big((\sigma_i + \sigma_e)\nabla u_e\big) = -div(\sigma_i \nabla v_m) & in\, \Omega \times]0, T[\\ \partial_t w + g(v_m, w) = 0 & in\, \Omega \times]0, T[\\ \sigma_i \nabla v_m . n = -\sigma_i \nabla u_e . n & on \Sigma \times]0, T[\\ (\sigma_i + \sigma_e)\nabla u_e . n = -\sigma_i \nabla v_m . n & on \Sigma \times]0, T[\end{cases}$$

(1)

Where A_m is defined to be the surface to volume ratio of the cell membrane, C_m the membrane capacitance, I_{ion} is the current due to the ion exchange. $g(v_m, w)$ is a function of fields having the same dimension of w which is the concentrations of different chemical , n is the outward unit normal on Ω , and $\Sigma = \Gamma_{endo} \cup \Gamma_{epi}$, Γ_{endo} internal boundaries in the rated endocardia and Γ_{epi} an outer boundaries, epicedial [19].

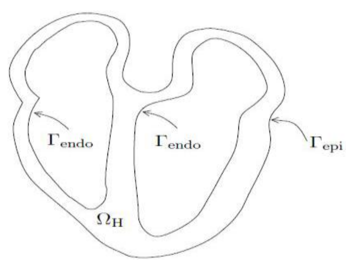

Figure. 1 Cardiac field distribution.

This model has been used by several researchers like Nielsen, Boulakia ,Henriquez and others they solved it by many methods we well rapprochement by ours.

2- FINITE VOLUME METHOD

The finite volume method is an approximate method for obtaining numerical solutions that has been used in several fields such as fluid mechanical engineering, heat transfer, petroleum engineering, and others. This method is similar to the finite element method, as it can be used in random shapes, with or without a regulator. It also preserves the numerical flow from one cell to another, making it an attractive way to model problems related to flow, such as heat and mass. Let us assume that the form of the conservation equation(in one dimensional) is:

$$F_t(X,t) + divF(X,t) = f(X,t) \qquad (2)$$

at each point x and each time t where the conservation of q is to be written. In equation (2), t

denotes the time partial derivative of the entity within the parentheses, div represents the space divergence operator: $divF = \partial F_1/\partial x_1 + \cdots + \partial F_d/\partial x_d$, where $F = (F_1, \ldots, F_d)^t$ denotes a vector function depending on the space variable x and on the time t, x_i is the i-th space coordinate, for $i = 1, \ldots, d$, and d is the space dimension, i.e. $d = 1, 2$ or 3; the quantity F is a flux which expresses a transport mechanism of q; the "source term" f expresses a possible volumetric exchange, due for instance to chemical reactions between the conserved quantities.[20]

It can be considered a special case that combines the finite element method and the finite difference method, The basic concept of FVM is to divide the domain into distinct (discrete) elements, known as

control volumes "CVs", where each control volume contains one node of the computation network. We divide the domain into discrete controllers whose global nodal point is defined by J and its neighbors in 2D geometry, East (E) and West (W) are adjacent nodes of theN obal grid J now also has neighbors to the north (N) and south (S) as shown in Fig. 2.

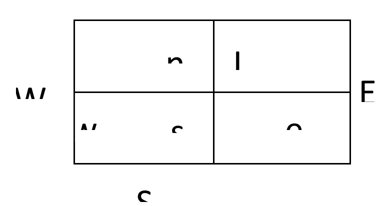

Fig.2 Control volume in the x-y direction.

The basic step of the finite volume method is integration equation on the control volume to obtain a separate equation at every point, The first-order forward difference approach can typically be used to estimate the time derivative, as shown below:

$$\frac{\partial q}{\partial t} = \frac{q_i^{n+1} - q_i^{n}}{\Delta t}$$
(3)

Where q is the unknown scalar quantity, and t ∈ (0;T), the approximation to the cell average of q over control volume Ω_i at time $t_n$and t_{n+1} is represented by

$$q_i^{\mathbf{n+1}} = \frac{1}{|\Omega_i|}\left(\int_{\Omega_i} q(x, y, t^{n+1})\right)dx$$
(4)

where $|\Omega_i|$ is the measure of Ω_i, For an arbitrary control volume, the flux integral over $\partial\Omega_i$ could be approximated by the summation of fluxes passing through all the adjacent cell faces.

The finite volume method can be easily generalized to irregular rectangular or parallel grids, that is, the method can be expanded to be two- or three-dimensional, but by taking Newman and Dirichlet conditions into account, let $q_t(x, t) = 0$ so equation (2) well be as

$$divF(x, y, t) = f(x, y, t) \qquad , (x, y, t) \in \Omega$$
(5)

And let K_{ij} ba an admissible mesh of $(0, l) \times (0, l)$ where $i = 1, \dots, N_1, j = 1, \dots, N_2$ and

$k_{i,j} = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right] \times \left[y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}\right]$ such that $x_{i-\frac{1}{2}} < x_i < x_{i+\frac{1}{2}}$ for $i = 1, \dots, N_1;\ x_0 = 0\ , x_{N_i+1} = l$

$y_{j-\frac{1}{2}} < y_j < y_{j+\frac{1}{2}}$ for $j = 1, \dots, N_2;\ y_0 = 0\ , y_{N_2+1} = l$.

The finite volyme method is found by integration over each control volume $k_{i,j}$ which be as $\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} F_x\left(x_{i+\frac{1}{2}}, y\right)dy + \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} F_x\left(x_{i-\frac{1}{2}}, y\right)dy + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} F_y\left(x, y_{j+\frac{1}{2}}\right)dx + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} F_y\left(x, y_{j-\frac{1}{2}}\right)dx = \int_{k_{i,j}} f(x, y)dxdy$
(6)

We can write it as anumerical scheme

$F_{i+\frac{1}{2},j} + F_{i-\frac{1}{2},j} + F_{i,j+\frac{1}{2}} + F_{i,j-\frac{1}{2}} = h_{i,j}\, f_{i,j} \quad ; \forall(i, j) \in \{i = 1, \dots, N_1\} \times \{j = 1, \dots, N_2\}$ where

$h_{i,j} = h_i \times k_j$, $f_{i,j}$ the mean value over $k_{i,j}$ [21].

If we take the equation (2) as the formella

$$q_t(x, y, t) = div(\nabla F(x, y, t)) \quad (7)$$

Then the integration by FVM over control volume Ω well be

$$\iint \frac{\partial q}{\partial t} d\Omega = \iint div(\nabla F)\, d\Omega \quad (8)$$

Equation (7) can be written as

$$\frac{\partial q}{\partial t} \Delta v = \sum (\nabla F)_u \,.\, n_u \,.\, A_u \quad (9)$$

Where Δv is the control volume, n_u is outward unit normal and A_u the area of the face. By applying (9) with control volume we get

$$\frac{\partial q}{\partial t} \Delta x \Delta y = \left[\left(\left(\frac{\partial q}{\partial x}\right)_e - \left(\frac{\partial q}{\partial x}\right)_w \right) \Delta y + \left(\left(\frac{\partial q}{\partial y}\right)_n - \left(\frac{\partial q}{\partial y}\right)_s \right) \Delta x \right] \quad (10)$$

By using forward difference, we get

$$q_{i,j}{}^{n+1} = q_{i,j}{}^{n} - \Delta t \left[\left(\frac{q_{i+1,j}{}^{n} - q_{i,j}{}^{n}}{\Delta x^2} - \frac{q_{i,j}{}^{n} - q_{i-1,j}{}^{n}}{\Delta x^2} \right) + \left(\frac{q_{i,j+1}{}^{n} - q_{i,j}{}^{n}}{\Delta x^2} - \frac{q_{i,j}{}^{n} - q_{i,j-1}{}^{n}}{\Delta x^2} \right) \right] \quad (11)$$

For the control volume at the botton

$$q_{i,j}{}^{n+1} = q_{i,j}{}^{n} - \Delta t \left[\left(\frac{q_{i+1,j}{}^{n} - q_{i,j}{}^{n}}{\Delta x^2} - 2\frac{q_{i,j}{}^{n} - qL}{\Delta x^2} \right) + \left(\frac{q_{i,j+1}{}^{n} - q_{i,j}{}^{n}}{\Delta x^2} - 2\frac{q_{i,j}{}^{n} - qF}{\Delta x^2} \right) \right] \quad (12)$$

Where qL and qF are the boundary values at the left and front of the point.[22]

3- NEW ITERATION METHOD

A semi-analytical numerical method (NIM) used to find solutions of second-order nonlinear differential equations was devised by Varsha Daftardar-Gejji and Hossein Jafari[23], Sometimes the equation contains both linear and nonlinear parts that can be written as

$u(x,y,t) = N\big(u(x,y,t)\big) + r(x,y,t)$
(13)

Where N is anonlinear operator, r the known function , u is the unknown function and x, y, t are denoted indepent variables. The approximate solution for equation (13) is a series solution represented as $u(x,y,t) = \sum_{i=0}^{\infty} u_i(x,y,t)$
(14)

Applying the series form (14) on the nonlinear operator N , we cam obtain

$\sum_{i=0}^{\infty} u_i(x,y,t) = N(u_0) + r(x,y,t) + \sum_{i=1}^{\infty}\{N(\sum_{\hat{\imath}=0}^{i} u_{\hat{\imath}}(x,y,t)) - N(\sum_{\hat{\imath}=0}^{i-1} u_{\hat{\imath}}(x,y,t))\}$ (15)

The terms of NIM will be as[24]

$$
\begin{aligned}
u_0 &= r(x,y,t) \\
u_1 &= N(u_0) \\
u_2 &= N(u_0 + u_1) - N(u_0) \\
&\vdots \\
u_{j+1} &= N\big(u_0 + \cdots + u_j\big) - N\big(u_0 + \cdots + u_{j-1}\big) \qquad j = 1,2,\ldots
\end{aligned}
$$
(16)

The FVM will be combined with the NIM in nonlinear limits to improve the numerical solution of the Finite Volume Method.

4- Applications

Solving the system (1) by FVM we get

$$
\begin{cases}
v_{ij}^{n+1} = v_{ij}^{n} - \frac{\Delta t}{C_m} I_{ij}^{n} + \frac{\sigma_i \Delta t}{A_m C_m}\left({u_i}_{xx}^{\ n} + {u_e}_{xx}^{\ n} + {u_i}_{yy}^{\ n} + {u_e}_{yy}^{\ n}\right) + \frac{\sigma_i \Delta t}{A_m C_m}\left({u_e}_{xx}^{\ n} + {u_e}_{yy}^{\ n}\right) \\
\sigma_i {u_i}_{xx}^{\ n} + \sigma_e {u_e}_{xx}^{\ n} + \sigma_i {u_i}_{yy}^{\ n} + \sigma_e {u_e}_{yy}^{\ n} = 0 \\
w_{ij}^{n+1} = w_{ij}^{n} - \Delta t {g_{ij}}^{n} \\
\begin{cases} \sigma_i {u_i}_{xx}^{\ n} = 0 \\ \sigma_i {u_i}_{yy}^{\ n} = 0 \end{cases} \\
\begin{cases} \sigma_i {u_i}_{xx}^{\ n} + \sigma_e {u_e}_{xx}^{\ n} = 0 \\ \sigma_i {u_i}_{yy}^{\ n} + \sigma_e {u_e}_{yy}^{\ n} = 0 \end{cases}
\end{cases}
$$
(17)

In this paper we just take two cases of approximation to system (1), if we divid the heart as internal and external parts only and by substituting the second equation into the first and third equations of the system (1), we get the following system $in\ \Omega \times]0, T[$

$$\begin{cases} \boldsymbol{A_m}\left(\boldsymbol{C_m}\frac{\partial \boldsymbol{v_m}}{\partial \boldsymbol{t}} + \boldsymbol{I_{ion}}(\boldsymbol{v_m}, \boldsymbol{w})\right) + \boldsymbol{div}(\boldsymbol{\sigma_e}\,\boldsymbol{\nabla}\,\boldsymbol{u_e}) = \boldsymbol{0} & \boldsymbol{in}\ \Omega \times]\boldsymbol{0}, \boldsymbol{T}[\\ \boldsymbol{\partial_t w} + \boldsymbol{g}(\boldsymbol{v_m}, \boldsymbol{w}) = \boldsymbol{0} & \boldsymbol{in}\ \Omega \times]\boldsymbol{0}, \boldsymbol{T}[\end{cases} \quad (18)$$

and on the boundaries of the heart in figure (1), the system (1) will be as:

$$\begin{cases} A_m\left(C_m\frac{\partial v_m}{\partial t} + I_{ion}(v_m, w)\right) = 0 & on\ \Sigma \times]0, T[\\ \frac{\partial w}{\partial t} + g(v_m, w) = 0 & on\ \Sigma \times]0, T[\end{cases} \quad (19)$$

with the initial conditions $v_m^0(0, \mathrm{x}, \mathrm{y}) = v_0(X), w_m^0(0, \mathrm{x}, \mathrm{y}) = w_0(X)$.[15].This results found by using MATLAB and for Specific values of x and y.

Case (1): When we use $\boldsymbol{I_{ion}}$ and $\boldsymbol{g}$ as linear functions, the systems (18) and (19) can be solved via FVM with the condition:

$$I_{ion}(v_m, w) = g(v_m, w) = v_m - w \quad (20)$$

Then

$$\begin{cases} \boldsymbol{v_{ij}^{n+1}} = \boldsymbol{v_{ij}^{n}} - \frac{\Delta \boldsymbol{t}}{\boldsymbol{C_m}}\boldsymbol{I_{ij}^{n}} - \frac{\boldsymbol{\sigma_e}\Delta \boldsymbol{t}}{\boldsymbol{A_m C_m}}(\boldsymbol{u_{e_{xx}}}^{\boldsymbol{n}} + \boldsymbol{u_{e_{yy}}}^{\boldsymbol{n}}) & \boldsymbol{in}\ \Omega \times]\boldsymbol{0}, \boldsymbol{T}[\\ \boldsymbol{w_{ij}^{n+1}} = \boldsymbol{w_{ij}^{n}} - \Delta \boldsymbol{t}(\boldsymbol{v_{ij}}^{\boldsymbol{n}} - \boldsymbol{w_{ij}}^{\boldsymbol{n}}) & \boldsymbol{in}\ \Omega \times]\boldsymbol{0}, \boldsymbol{T}[\end{cases} \quad (21)$$

Where $u_{e_{xx}}{}^{n} = u_{e_{yy}}{}^{n} = \frac{u^n{}_{i+1j} - u^n{}_{ij}}{\Delta x^2} + 2\frac{u^n{}_{ij} - ul^n}{\Delta x^2} + \frac{u^n{}_{ij+1} - u^n{}_{ij}}{\Delta y^2} + 2\frac{u^n{}_{ij} - u^n f}{\Delta y^2}$,

$\Delta x = \Delta y = 0.1$

$\Delta t = 0.0001, ul = (y^2 - 1)^3 e^{-t}$ at $x = 0$, $uf = (x^2 - 1)^3 e^{-t}$ at $y = 0$.

And the boundary conditions of system (19) becomes:

$$\begin{cases} v_{ij}^{n+1} = v_{ij}^{n} - \frac{\Delta t}{C_m} I_{ij}^{n} & on\ \Sigma \times]0,T[\\ w_{ij}^{n+1} = w_{ij}^{n} - \Delta t({v_{ij}}^{n} - {w_{ij}}^{n}) & on\ \Sigma \times]0,T[\end{cases}$$
(22)

Where the exact solution for system (1) with values ${I^n}_{ij} = {v^n}_{ij} - {w^n}_{ij}$; $A_m = 1$; $C_m = 1$; $\sigma_e = 1$ $Vm = ((x^2 + y^2 - 1)^3 - x^2y^3)e^{-t}$; $w = 0$
(23)

At the boundaries of the heart, the boundary conditions towards x are used when $x = x_0$ with $y = y_0,\ y = y_L$ in system (22), while for the boundaries of the heart towards y, then the boundary conditions towards y are used when $y = y_0$ with $x = x_0,\ x = x_L$ in system (22).

TABLE 1. Demonstrates comparison of numerical solution using FVM with the exact solution when $n = 10, x \in [0,1]; y \in [0,1]; t = 0.001$

x	y	V_{exact}	V_{FVM}	$Errore_{FVM}$
0	0	-0.42285348	-0.42285348	0.0×10^{-8}
0.1	0.1	-0.41869382	-0.41869376	0.5×10^{-7}
0.2	0.2	-0.38698149	-0.38698062	0.87×10^{-6}
0.3	0.3	-0.33017554	-0.33017158	0.396×10^{-5}
0.4	0.4	-0.25940536	-0.25939453	0.1083×10^{-7}
0.5	0.5	-0.18880930	-0.18878630	0.2301×10^{-4}
0.6	0.6	-0.13337690	-0.13333490	0.4200×10^{-4}
0.7	0.7	-0.10607169	-0.10600238	0.6931×10^{-4}
0.8	0.8	-0.11423485	-0.11412837	1.0648×10^{-4}
0.9	0.9	-0.15526948	-0.15511449	$1.5500\times$

	10^{-4}
MSE	$2.73435\times e^{-7}$

noting that the approximation solution in table (1) is significantly improved when Segmentation the domain solution area of the heart, bringing the absolute error to $\mathbf{10^{-7}}$.

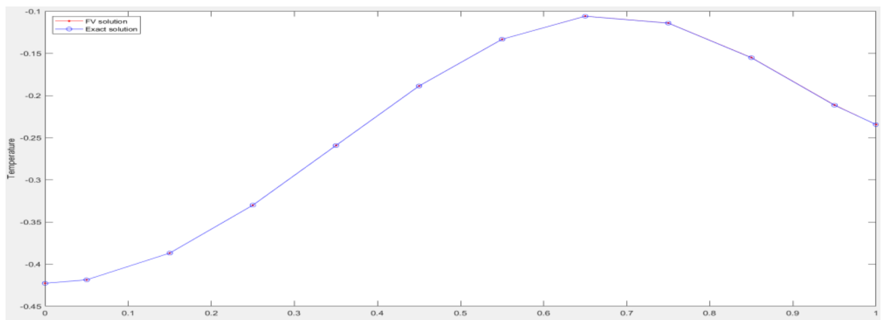

Figure 3: Solutions path for system (1) at the boundary conditions using FVM.

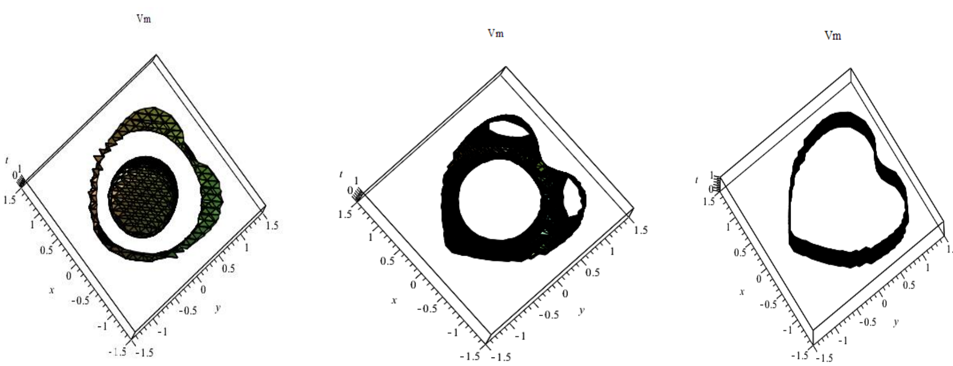

Figure 4: exact solution

Case 2: When we use $\boldsymbol{I_{ion}}$ and $\boldsymbol{g}$ as a nonlinear function as an exact solution we get the following system $\boldsymbol{in}\ \Omega \times]\mathbf{0}, \boldsymbol{T}[$, the systems (18) and (19) can be solved via FVM with the condition:

$I_{ion}(v_m, w) = w(1 - v_m)$, $\quad g(v_m, w) = \mathrm{f}(\mathrm{x}, \mathrm{y}, \mathrm{t})\, w \quad$ and $\quad f(x, y, t) = \frac{-5}{3}\frac{\sqrt{6}\, e^{\frac{1}{6}\sqrt{6}x+\frac{1}{6}\sqrt{6}y-\frac{5}{6}t}}{(1+\sqrt{6}\, e^{\frac{1}{6}\sqrt{6}x+\frac{1}{6}\sqrt{6}y-\frac{5}{6}t})}$ (24)

Then

$$\begin{cases} v_{ij}^{n+1} = v_{ij}^{n} - \frac{\Delta t}{C_m} I_{ij}^{n} - \frac{\sigma_e \Delta t}{A_m C_m} ({u_{e_{xx}}}^{n} + {u_{e_{yy}}}^{n}) & in\ \Omega \times]0, T[\\ w_{ij}^{n+1} = w_{ij}^{n} - \Delta t {g_{ij}}^{n} & in\ \Omega \times]0, T[\end{cases}$$

(25)

Where $\quad {u_{e_{xx}}}^{n} = {u_{e_{yy}}}^{n} = \frac{{u^n}_{i+1j} - {u^n}_{ij}}{\Delta x^2} + 2\frac{{u^n}_{ij} - ul^n}{\Delta x^2} + \frac{{u^n}_{ij+1} - {u^n}_{ij}}{\Delta y^2} + 2\frac{{u^n}_{ij} - u^n f}{\Delta y^2}$, $\Delta x = \Delta y = 0.1$

$\Delta t = 0.0001$, $ul = (1 + e^{\sqrt{\frac{1}{6}}y - (-2)\frac{5}{6}t})^{-2}$ at $x = 0$, $uf = (1 + e^{\sqrt{\frac{1}{6}}x - (-2)\frac{5}{6}t})^{-2}$ at $y = 0$.

And the boundary conditions of system (19) becomes:

$$\begin{cases} v_{ij}^{n+1} = v_{ij}^{n} - \frac{\Delta t}{C_m} I_{ij}^{n} & on\ \Sigma \times]0, T[\\ w_{ij}^{n+1} = w_{ij}^{n} - \Delta \mathrm{t} {g_{ij}}^{n} & on\ \Sigma \times]0, T[\end{cases}$$

(26)

Where the exact solution for system (1) with values ${I^n}_{ij} = 2{w^n}_{ij}(1 - {v^n}_{ij})$; $A_m = 1$;$C_m = 1$;$\sigma_e = 2$ ${g_{ij}}^{n} = {f_{ij}}^{n}$; ${w_{ij}}^{n}$ and $Vm = (1 + e^{\sqrt{\frac{1}{6}}x + \sqrt{\frac{1}{6}}y - (-2)\frac{5}{6}t})^{-2}$ (27)

At the boundaries of the heart, the boundary conditions towards x are used when $x = x_0$ with $y = y_0$, $y = y_L$ in system (26), while for the boundaries of the heart towards y, then the boundary conditions towards y are used when $y = y_0$ with $x = x_0$, $x = x_L$ in system (26).

TABLE 2. Demonstrates comparison of numerical solution using FVM with the exact solution when $n = 10, x \in [0,1]; y \in [0,1]; t = 0.001$

x	y	V_{exact}	V_{FVM}	$Errore_{FVM}$
0	0	0.20139163	0.20139163	$0.0\times \mathbf{10^{-8}}$
0.1	0.1	0.19689054	0.19915887	$2.26833\times \mathbf{10^{-3}}$
0.2	0.2	0.18807194	0.18885302	$7.8108\times \mathbf{10^{-4}}$
0.3	0.3	0.17950135	0.18019363	$6.9228\times \mathbf{10^{-4}}$
0.4	0.4	0.17118199	0.17185178	$6.6979\times \mathbf{10^{-4}}$
0.5	0.5	0.16311631	0.16376528	$6.4897\times \mathbf{10^{-4}}$
0.6	0.6	0.15530601	0.15593409	$6.2807\times \mathbf{10^{-4}}$
0.7	0.7	0.14775207	0.14835922	$6.0715\times \mathbf{10^{-4}}$
0.8	0.8	0.14045475	0.14104231	$5.8756\times \mathbf{10^{-4}}$
0.9	0.9	0.13341361	0.13402986	$6.1625\times \mathbf{10^{-4}}$
MSE				$5.62422\ e^{-6}$

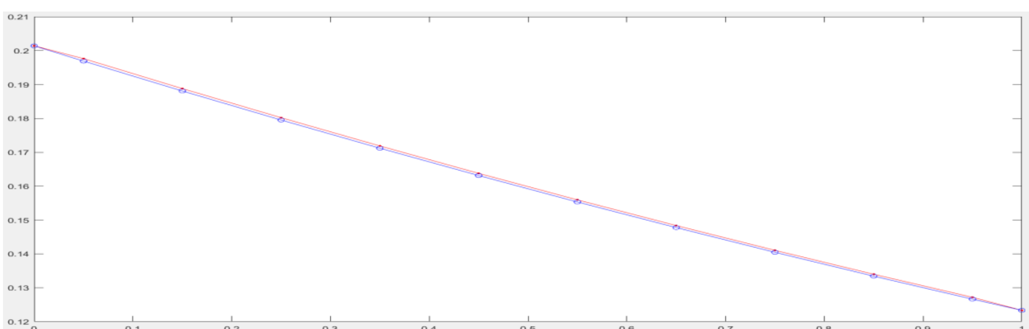

Figure 5: Solutions path for system (1) at the boundary conditions using FVM.

TABLE 3. Demonstrates comparison of numerical solution using FVM

with the exact solution when $n = 10, x \in [0,1]; y \in [0,1]; t = 0.001$

x	y	W_{exact}	W_{FVM}	$Errore_{FVM}$
0	0	0.20139163	0.20139163	$0.0\times \mathbf{10^{-8}}$
0.1	0.1	0.19689054	0.19762098	$2.26833\times \mathbf{10^{-3}}$
0.2	0.2	0.18807194	0.18878229	$7.8108\times \mathbf{10^{-4}}$
0.3	0.3	0.17950135	0.18019131	$6.9228\times \mathbf{10^{-4}}$
0.4	0.4	0.17118199	0.17185134	$6.6979\times \mathbf{10^{-4}}$
0.5	0.5	0.16311631	0.16376487	$6.4897\times \mathbf{10^{-4}}$
0.6	0.6	0.15530601	0.15593367	$6.2807\times \mathbf{10^{-4}}$
0.7	0.7	0.14775207	0.14835878	$6.0715\times \mathbf{10^{-4}}$
0.8	0.8	0.14045475	0.14104051	$5.8756\times \mathbf{10^{-4}}$
0.9	0.9	0.13341361	0.13397848	$6.1625\times \mathbf{10^{-4}}$
MSE				$5.62422\ e^{-6}$

noting that the approximation solution in table (2) and table (3) are significantly improved when Segmentation the domain solution area of the heart, bringing the absolute error to $\mathbf{10^{-6}}$.

Case ٣: In this case, we will improve the results we obtained from case٢ by solving the systems (18) and (19) using FVM, and by solving the nonlinear term I_{ion} using NIM according to the function

$$I_{ion_{i,j}}{}^{n} = \sum_{n=1}^{\infty}\left(\sum_{m=0}^{n} I_{ion_{i,j}}{}^{m} - \sum_{m=0}^{n-1} I_{ion_{i,j}}{}^{m}\right)$$
(28)

with the condition:

$I_{ion}(v_m, w) = w(1 - v_m)$, $g(v_m, w) = \mathrm{f(x, y, t)}\, w$ and $f(x, y, t) = \frac{-5}{3} \frac{\sqrt{6}\, e^{\frac{1}{6}\sqrt{6}x+\frac{1}{6}\sqrt{6}y-\frac{5}{6}t}}{(1+\sqrt{6}\, e^{\frac{1}{6}\sqrt{6}x+\frac{1}{6}\sqrt{6}y-\frac{5}{6}t})}$ (29)

$$\begin{cases} v_{ij}^{n+1} = v_{ij}^{n} - \frac{\Delta t}{C_m} \sum_{n=1}^{\infty} \left(\sum_{m=0}^{n} I_{i,j}{}^{m} - \sum_{m=0}^{n-1} I_{i,j}{}^{m} \right) - \frac{\sigma_e \Delta t}{A_m C_m} (u_{e_{xx}}{}^{n} + u_{e_{yy}}{}^{n}) & in\ \Omega \times]0, T[\\ w_{ij}^{n+1} = w_{ij}^{n} - \Delta t g_{ij}{}^{n} & in\ \Omega \times]0, T[\end{cases}$$
… (30)

Where $u_{e_{xx}}{}^{n} = u_{e_{yy}}{}^{n} = \frac{u^{n}{}_{i+1j} - u^{n}{}_{ij}}{\Delta x^2} + 2\frac{u^{n}{}_{ij} - ul^{n}}{\Delta x^2} + \frac{u^{n}{}_{ij+1} - u^{n}{}_{ij}}{\Delta y^2} + 2\frac{u^{n}{}_{ij} - u^{n}f}{\Delta y^2}$,
$\Delta x = \Delta y = 0.1$

$\Delta t = 0.0001$, $ul = (1 + e^{\sqrt{\frac{1}{6}}y - (-2)\frac{5}{6}t})^{-2}$ at x, $uf = (1 + e^{\sqrt{\frac{1}{6}}x - (-2)\frac{5}{6}t})^{-2}$ at $y = 0$.

And the boundary conditions of system (19) becomes:

$$\begin{cases} v_{ij}^{n+1} = v_{ij}^{n} - \frac{\Delta t}{C_m} \sum_{n=1}^{\infty} \left(\sum_{m=0}^{n} I_{i,j}{}^{m} - \sum_{m=0}^{n-1} I_{i,j}{}^{m} \right) & on\ \Sigma \times]0, T[\\ w_{ij}^{n+1} = w_{ij}^{n} - \Delta \mathrm{t} g_{ij}{}^{n} & on\ \Sigma \times]0, T[\end{cases}$$
(31)

Where the exact solution for system (1) with values $I^{n}{}_{ij} = 2w^{n}{}_{ij}(1 - v^{n}{}_{ij})$; $A_m = 1$;$C_m = 1$;$\sigma_e = 2$ $g_{ij}{}^{n} = f_{ij}{}^{n}$; $w_{ij}{}^{n}$ and $Vm = (1 + e^{\sqrt{\frac{1}{6}}x + \sqrt{\frac{1}{6}}y - (-2)\frac{5}{6}t})^{-2}$ (32)

the values well be as following table:

TABLE 4. Demonstrates comparison of numerical solution using FVM

vai NIM with the exact solution when $n = 10, x \in [0,1]; y \in [0,1]; t = 0.001$

x	y	V_{exact}	V_{NIM}	$Errore_{NIM}$
0	0	0.20139163	0.20139163	$0.0\times \mathbf{10^{-8}}$
0.1	0.1	0.19689054	0.19855043	$1.6599\times \mathbf{10^{-3}}$
0.2	0.2	0.18807194	0.18824813	$1.762\times \mathbf{10^{-4}}$
0.3	0.3	0.17950135	0.17963937	$1.3802\times \mathbf{10^{-4}}$
0.4	0.4	0.17118199	0.17134580	$1.638\times \mathbf{10^{-4}}$
0.5	0.5	0.16311631	0.16330414	$1.8783\times \mathbf{10^{-4}}$
0.6	0.6	0.15530601	0.15551450	$2.0849\times \mathbf{10^{-4}}$
0.7	0.7	0.14775207	0.14797806	$2.2599\times \mathbf{10^{-4}}$
0.8	0.8	0.14045475	0.14069662	$2.4188\times \mathbf{10^{-4}}$
0.9	0.9	0.13341361	0.13371731	$3.037\times \mathbf{10^{-4}}$
MSE				$1.09284\ e^{-6}$

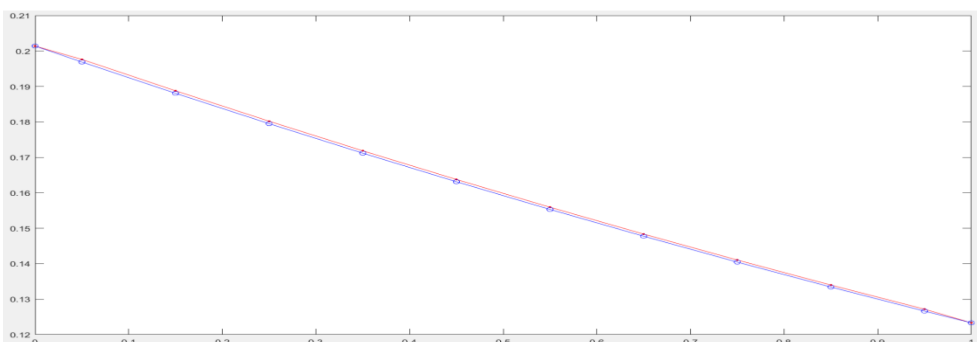

Figure 6: Solutions path for system (1) using modified FVM by NIM.

TABLE 5. Demonstrates comparison of numerical solution using FVM

vai NIM with the exact solution when $n = 10, x \in [0,1]; y \in [0,1]; t = 0.001$

x	y	W_{exact}	W_{FVM}	$Errore_{FVM}$
0	0	0.20139163	0.20139163	$0.0\times \mathbf{10^{-8}}$
0.1	0.1	0.19689054	0.19762098	$1.6599\times \mathbf{10^{-3}}$
0.2	0.2	0.18807194	0.18878229	$1.762\times \mathbf{10^{-4}}$
0.3	0.3	0.17950135	0.18019131	$1.3802\times \mathbf{10^{-4}}$
0.4	0.4	0.17118199	0.17185134	$1.6381\times \mathbf{10^{-4}}$
0.5	0.5	0.16311631	0.16376487	$1.8783\times \mathbf{10^{-4}}$
0.6	0.6	0.15530601	0.15593367	$2.0849\times \mathbf{10^{-4}}$
0.7	0.7	0.14775207	0.14835878	$2.2599\times \mathbf{10^{-4}}$
0.8	0.8	0.14045475	0.14104051	$2.4188\times \mathbf{10^{-4}}$
0.9	0.9	0.13341361	0.13397848	$3.037\times \mathbf{10^{-4}}$
MSE				$1.09284\ e^{-6}$

noting that the approximation solution in table (4) and table (5) are significantly improved when Segmentation the domain solution area of the heart, bringing the absolute error to $\mathbf{10^{-6}}$.

5- Discussion

We note that the numerical results using a linear initial condition are good, with a small error rate 10^{-9}. As for the second case, i.e. using a non-linear initial condition, the values of the variables are better, with an error rate 10^{-6}. While using the new iteration method to solve the nonlinear limit and then using the finite volume method for the solution, we notice that the numerical solutions were closer to the exact solutions, meaning that we are approaching the real solutions faster and with fewer operations.

6- CONCLUSION

Any model is usually solved directly due to easy access to initial conditions and boundary values, unlike inverse problems in which one of the initial conditions or boundary values is missing, arriving at the solution becomes a challenge, and since numerical simulation of the heart or any other organ can give Wrong results if not verified properly, by examining the tables, we notice that if the function g, I_{ion} used as a linear function, the error rate increases gradually after a period of time, that is, it begins almost identically and then begins to converge after several iterations. However, in the second case, when a non-linear function is used, it begins with a large error rate and then begins after several repetitions. The repetitions increase, that is, they converge after a period of time. This does not mean that we cannot rely on this method to find solutions,that the results can be improved by solving the nonlinear term with other numerical methods to obtain a lower error rate, as we observe in the third case, knowing that both cases give a small error rate, so it can be relied upon to perform medical intervention in a timely manner.

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