On the Independent, Restrained, Total and Connected Domination Number of Musical Graphs

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Abstract:

A set S of vertices in a graph $G = (V, E)$ called a dominating set if every vertex $v \in V$ either is in S or is adjacent to at least one vertex in S .The domination number of a graph G denoted by $\gamma(G)$ is the minimum size of the dominating sets of G. In this paper we studied the domination number, independent, restrained, total and connected domination number in Musical graphs.

حول العدد المهيمن، المستقل، المقيد، الكلي والمتصل للبيانات الموسيقية

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ملخص البحث:

يقال لمجموعة S من رؤوس البيان $G = (V, E)$ أنها مجموعة مهيمنة Dominating set إذا كان لكل رأس من رؤوس المجموعة V إما هي ضمن المجموعة S أو يجاور رأس واحد في الأقل من الرؤوس التي هي في S. فيعرف بـذلك العـدد المهـيمن Domination number يرمز له $\gamma(G)$ بأنه حجم اصغر مجموعة ضمن المجموعات المهيمنة. في هذا البحث درسنا العدد المهيمن D والمستقل independent والمقيد restrained والكلي total والعدد المهيمن المتصل connected في البيان الموسيقي.

Introduction:

In this paper, we follow the notation of [6] and [7]. Specifically, let *G=(V,E)* be a graph with vertex set V and edge set E. We consider simple graphs that are undirected, un weighted and contain no loops or multiple edges. Moreover, the notations $< D >$ denotes the induced sub graph of *G* by the vertices of *D*.

A set $S \subseteq V$ is dominating set of *G* if every vertex not in *S* is adjacent to a vertex in *S*. The domination number of *G* denoted by $\gamma(G)$ is the minimum cardinality of dominating set of *G*. Also *G*[*S*] is the sub graph induced by *S*, and the cardinality of a set S denoted by $|S|$.

A dominating set *S* of the graph *G* is said to be independent if no two vertices of *S* are connected by an edge of *G*. The independent domination number of a graph *G*, denoted by $\gamma_i(G)$ is the minimum size of smallest independent domination set of *G*.[3]

A set $S \subseteq V$ is a restrained dominating set if every vertex not in S is adjacent to a vertex in *S* and to a vertex in $V - S$. Every graph has restrained dominating set since $V = S$ is such a set. The restrained domination number of *G* denoted by $\gamma_r(G)$ is the minimum cardinality restrained dominating set of *G*.[5]

A dominating set $D \subseteq V$ of a graph G is said to be connected dominating set if the induced sub graph $< D >$ is connected. The minimum cardinality of connected dominating set is the connected domination number $\gamma_c(G)$.[4]

A set S is total dominating set of *G* if $G[S]$ has no isolated vertex. The total domination number of *G* ,denoted by $\gamma_t(G)$ is the minimum size of a total dominating set of *G*.[10]

The concept of domination in graphs, with its many variations is now well studied in graph theory. The book by ***Chartrand and Lesniak*** [2] contains a chapter on domination.

To date many papers have been written on domination in graphs likes [1],[8],[9] and [13] .

In this section we need the following theorem and lemma:

Theorem A (see [10]): If *G* has no isolated vertices, then $\gamma(G) \leq \frac{|V(G)|}{2}$.

Lemma (see [11]): Let *S* be a total dominating set with this property that every $x \in V(G)$ is dominated by exactly one vertex of *S*, then *S* is a minimum total dominating set.

On domination number, independent and restrained domination number of $M_{2,m}$.

Definition1: The strong product of K_2 and C_m , $m \geq 3$ is denoted by $M_{2,m}$. For $3 \leq m \leq 12$, $M_{2,m}$ is called Musical graph (see Fig.1).[12].

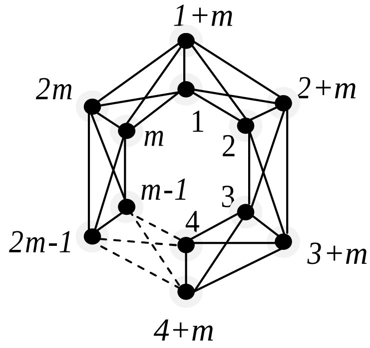

Fig.1.The graph $K_2 \boxtimes C_m (= M_{2,m})$

It is Known that $\gamma(C_m) = \left\lceil \frac{m}{3} \right\rceil$.

Proposition 2: For $m \geq 3$, $\gamma(M_{2,m}) = \left\lceil \frac{m}{3} \right\rceil$.

Proof: It is clear that each vertex i of C_m dominates $i-1$ and $i+1$ in C_m.

In $M_{2,m}$ vertex i also dominates $i+m$, $i+m-1$ and $i+m+1$.Therefore if S is a dominate set of C_m, then it is a dominate set of $M_{2,m}$.

Therefore $\gamma(M_{2,m}) = \gamma(C_m) = \left\lceil \frac{m}{3} \right\rceil$. ■

One can easily see that S is minimum domination set,where

$$S = \begin{cases} \{1,4,7,\dots,3k-2\}, if\ m = 3k \\ \{1,4,7,\dots,3k-2,3k\}, if\ m = 3k+1 \text{ or } 3k+2. \end{cases} \quad \dots (*)$$

And, $|S| = \left\lceil \frac{m}{3} \right\rceil$ and S is independent set in $M_{2,m}$.

Corollary 3: $\gamma_i(M_{2,m}) = \left\lceil \frac{m}{3} \right\rceil$

Proof: $\gamma_i(G) \geq \gamma(G)$. Thus, the proof is obvious .■

Corollary 4: $\gamma_r(M_{2,m}) = \left\lceil \frac{m}{3} \right\rceil$

Proof:

$$\gamma_r(M_{2,m}) \geq \gamma(M_{2,m}) = \left\lceil \frac{m}{3} \right\rceil. \quad \dots (2.4.1)$$

Consider the set S defined in (*). Because S is dominating set, then every vertex in $V(M_{2,m}) - S$ is adjacent to a vertex in S. Also, each vertex $M_{2,m}$ is adjacent to at least one vertex from the outer cycle $(m+1, m+2, \dots, 2m, m+1)$

Therefore, S is a restrained dominating set. Thus $\gamma_r(M_{2,m}) \leq |S| = \left\lceil \frac{m}{3} \right\rceil$. ... (2.4.2)

Hence, the proof from (2.4.1) and (2.4.2) .■

On the connected and total domination numbers of $M_{2,m}$:

In this section, we study the connected domination number and the total domination number of the Musical graph $M_{2,m}$.

We first given a special case of $\gamma_c(M_{2,3})$ in the following proposition then generalize for other chooses of *m* .

Proposition 1: $\gamma_c(M_{2,3}) = 2$

Proof: it is very easy to verify that the set of vertices $S = \{1,2\}$ is the connected domination set of $M_{2,3}$ and clearly S is minimum because there is no proper connected sub graph dominates $M_{2,3}$. ■

Theorem 2: for $m > 3$, $\gamma_c(M_{2,m}) = m - 2$

Proof: Let $S = \{i | i = 1,2, \dots, m-2\}$ be the connected dominating set of minimum size.

Therefore $\gamma_c(M_{2,m}) \leq |S| = m - 2$ (by theorem A).

The $< S >$ is a path , and dominate at most $4 + 2|S|$ vertices.

Thus, $4 + 2|S| \geq 2m$

This implies that $|S| \geq m - 2$

$\therefore \gamma_c(M_{2,m}) = m - 2.$ ■

Theorem 3: For $m \geq 3$

$$\gamma_t(M_{2,m}) = \begin{cases} \frac{m+1}{2} & \text{for } m \equiv 1 \ (mod\ 4) \\ 2\left\lceil \frac{m}{4} \right\rceil & \text{for other values of } m \geq 3 \end{cases}$$

Proof: Take $S = S_1 \cup S_2 \cup \{m\}$, where $S_1 = \{1 + 4k \,|\, k = 0,1,2, \dots, \left\lfloor \frac{m-2}{4} \right\rfloor\}$ and $S_2 = \{2 + 4k \,|\, k = 0,1,2, \dots, \left\lfloor \frac{m-2}{4} \right\rfloor\}$.

Then $|S| = 2\left\lfloor \frac{m-2}{4} \right\rfloor + 3.$

For $m = 4t + 1$, we get

$|S| = 2\left\lfloor \frac{4t-1}{4} \right\rfloor + 3 = 2\frac{4(t-1)}{4} + 3 = 2t + 1 = \frac{m+1}{2}$

Thus, for $m \equiv 1 \ (mod\ 4)$, $\gamma_t(M_{2,m}) \leq \frac{m+1}{2}$

Now, assume S' is total dominating minimum set, then $< S' >$ consists of independent edges with one P_3 or independent edges only (when $|S'| = \gamma_t(M_{2,m})$ is even).

Now, for $m \equiv 1 \ (mod\ 4)$, $< S' >$ must consist of $\frac{1}{2}(\gamma_t(M_{2,m}) - 3)$ independent edges and P_3 (a path of order 3).

Each edge of $< S' >$ dominates at most 8 vertices and P_3 dominates at most 10 vertices.

Thus, $8 \cdot \frac{1}{2}[\gamma_t(M_{2,m}) - 3] + 10 \geq 2m$

$\therefore \ 2\,\gamma_t(M_{2,m}) \geq m + 1$. This implies that $\gamma_t(M_{2,m}) \geq \frac{m+1}{2}$

Thus, for $m \equiv 1 \ (mod\ 4)$, we have $\gamma_t(M_{2,m}) = \frac{m+1}{2}$

Now, assume $m \not\equiv 1 \ (mod\ 4)$, and take $S = S_1 \cup S_2$.

Then $|S| = 2\left\lfloor \frac{m-2}{4} \right\rfloor + 2 = 2\left(\left\lfloor \frac{m-2}{4} \right\rfloor + 1\right) = 2\left\lceil \frac{m}{4} \right\rceil$.

$\therefore \gamma_t(M_{2,m}) \leq 2\left\lceil\frac{m}{4}\right\rceil$. … (3.3.1)

In this case, $< S' >$ consists of independent edges.

Therefore, $8 \cdot \frac{1}{2}\, \gamma_t(M_{2,m}) \geq 2m$

$$\gamma_t(M_{2,m}) \geq 2(\tfrac{m}{4})$$

Since $\gamma_t(M_{2,m})$ is even integer, then $\gamma_t(M_{2,m}) \geq 2\left\lceil\frac{m}{4}\right\rceil$. … (3.3.2)

From (3.3.1) and (3.3.2), we get

$\therefore \gamma_t(M_{2,m}) - 2\left\lceil\frac{m}{4}\right\rceil + 2$ for $m \not\equiv 1\ (mod\ 4)$. ■

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